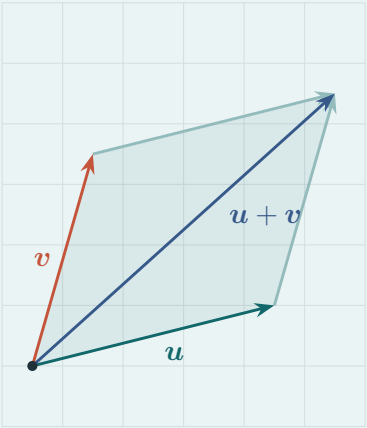
ANALYTIC GEOMETRY · LECTURE 1

Points, vectors and coordinates

Where is it, and how do we get there?



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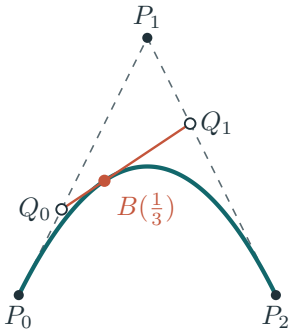
Two questions for today

YOUR TURN · a moving marker

A marker moves in a straight line from $A = (1, 1)$ to $B = (5, 3)$ at constant speed. Where is it after a quarter of the journey?

YOUR TURN · a smooth curve

A screen can only draw straight pieces and compute averages of points. How does it draw a smooth arc?



Both questions have one-line answers once we can compute with points and vectors.

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Points and vectors

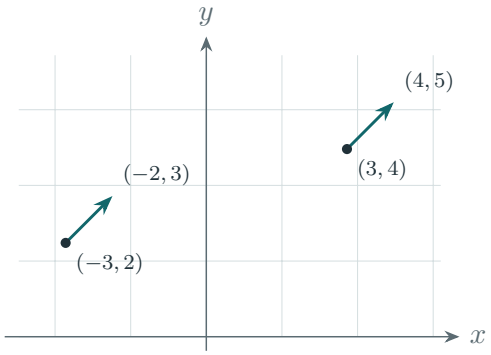
IN THIS LECTURE

1 Points and vectors
2 Coordinates
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4 Affine combinations
5 Interpolation and Bézier curves

A position is not a movement

- **Where is it?** A point P: a position.
- **How do we get there?** A vector: a displacement, “move this far, in this direction”.

From $(3, 4)$ to $(4, 5)$ and from $(-3, 2)$ to $(-2, 3)$:
four different points, *the same movement*.



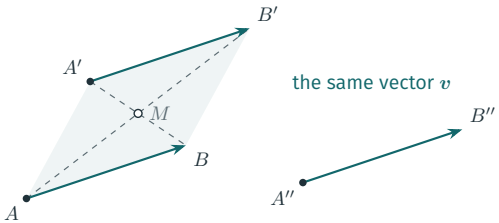
When do two arrows describe the same movement?

An *oriented segment* (A, B) is an arrow from A to B.

DEFINITION · equipollent segments

(A, B) and (A', B') are *equipollent* if $[AB']$ and $[A'B]$ have the same midpoint.

- In general position: $ABB'A'$ is a parallelogram (its diagonals bisect each other).
- The definition also covers arrows on one line and $A = B$.



Vectors

DEFINITION · vector

The *vector* $\overrightarrow{AB}$ is the class of all oriented segments equipollent to (A, B). Each of them *represents* the vector.

- zero vector $\mathbf{0} = \overrightarrow{AA}$; opposite $-\overrightarrow{AB} = \overrightarrow{BA}$;
- a nonzero vector has a length, a direction and a sense,
- …but no starting point.

PROPOSITION · placing a vector

For every point P and vector $\boldsymbol{v}$ there is exactly one Q with $\overrightarrow{PQ} = \boldsymbol{v}$. We write $Q = P + \boldsymbol{v}$.

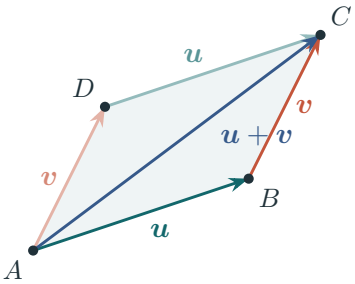
Why: if $\boldsymbol{v} = \overrightarrow{AB}$, then Q is the reflection of A in the midpoint of $[PB]$.

Adding vectors: the triangle and parallelogram rules

Put $B = A + \boldsymbol{u}$ and $C = B + \boldsymbol{v}$. Then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$:

$$\boxed{\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}} \qquad \text{(Chasles)}$$

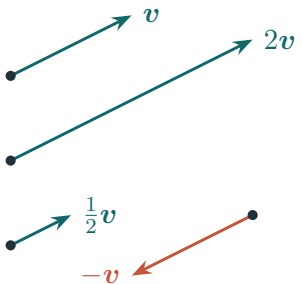
If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{AD}$ start at the same point, the sum is the diagonal $\overrightarrow{AC}$ of the parallelogram $ABCD$. The result does not depend on where we start: moving A translates the whole picture.



Multiplying a vector by a number

DEFINITION · scalar multiple

For $\lambda \in \mathbb{R}$: $\lambda \boldsymbol{v}$ is parallel to $\boldsymbol{v}$, its length is $|\lambda|$ times the length of $\boldsymbol{v}$; same sense if $\lambda > 0$, opposite sense if $\lambda < 0$. Also $0\,\boldsymbol{v} = \lambda\,\mathbf{0} = \mathbf{0}$.



Subtraction: $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$, so

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}.$$

The rules of vector algebra

For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and scalars λ, μ:

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} \qquad \lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v}$$

$$(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) \qquad (\lambda + \mu)\boldsymbol{u} = \lambda\boldsymbol{u} + \mu\boldsymbol{u}$$

$$\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u} \qquad \lambda(\mu\boldsymbol{u}) = (\lambda\mu)\boldsymbol{u}$$

$$\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u}$$

Commutativity is the parallelogram rule read in two ways; $\lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v}$ is Thales' theorem. In coordinates every rule becomes a rule about real numbers.

We may calculate with vectors as with ordinary algebraic expressions: the vectors of the plane, or of space, form a real vector space.

Collinear vectors and points on a line

DEFINITION · collinear vectors

$\boldsymbol{u}$ and $\boldsymbol{v}$ are *collinear* if one of them is a scalar multiple of the other. ($\mathbf{0}$ is collinear with every vector.)

PROPOSITION · points on a line

Let $A \neq B$. A point P lies on the line AB if and only if

$$\overrightarrow{AP} = t\,\overrightarrow{AB} \quad \text{for some } t \in \mathbb{R},$$

and this t is unique.

Uniqueness: $t\,\overrightarrow{AB} = s\,\overrightarrow{AB}$ gives $(t-s)\overrightarrow{AB} = \mathbf{0}$, and $\overrightarrow{AB} \neq \mathbf{0}$ forces $t = s$.

Comparing coefficients

PROPOSITION

If $\boldsymbol{u}, \boldsymbol{v}$ are *not* collinear and $\alpha\boldsymbol{u} + \beta\boldsymbol{v} = \alpha'\boldsymbol{u} + \beta'\boldsymbol{v}$, then $\alpha = \alpha'$ and $\beta = \beta'$.

Proof. $(\alpha - \alpha')\boldsymbol{u} = (\beta' - \beta)\boldsymbol{v}$.

- If $\alpha \neq \alpha'$: $\boldsymbol{u} = \frac{\beta' - \beta}{\alpha - \alpha'}\,\boldsymbol{v}$, so $\boldsymbol{u}, \boldsymbol{v}$ are collinear. Impossible.
- So $\alpha = \alpha'$, then $(\beta' - \beta)\boldsymbol{v} = \boldsymbol{0}$ with $\boldsymbol{v} \neq \boldsymbol{0}$: $\beta = \beta'$.

APPLICATION · the standard trick

To find where two lines meet, write the meeting point in two ways and compare coefficients.

→ board problem 1.B4

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Coordinates

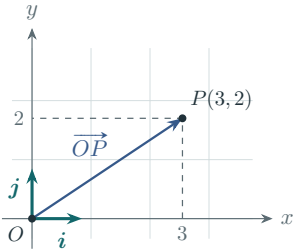
IN THIS LECTURE

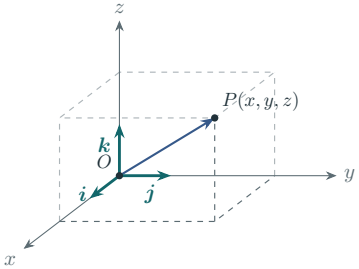
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Cartesian frames

DEFINITION · Cartesian frame

An origin O and perpendicular vectors of length 1: $\boldsymbol{i}, \boldsymbol{j}$ in the plane, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in space (right-handed).





Right-handed: thumb $\boldsymbol{i}$, index $\boldsymbol{j}$, middle finger $\boldsymbol{k}$. The picture of space is schematic.

Coordinates of vectors and of points

PROPOSITION · coordinates

Every vector can be written *in exactly one way* as

$$\boldsymbol{v} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}, \qquad \text{and we write } \boldsymbol{v} = (x, y, z).$$

- Existence: drop perpendiculars to the axes (a rectangular box).
- Uniqueness: compare coefficients.
- The *position vector* of P is $\boldsymbol{p} = \overrightarrow{OP}$; the coordinates of P are those of $\boldsymbol{p}$.
- Convention: $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, …

Computing with coordinates

Coordinate by coordinate:

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3),$$
$$\lambda \boldsymbol{u} = (\lambda u_1,\, \lambda u_2,\, \lambda u_3).$$

The most used formula of the course:

$$\boxed{\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}}$$

Proof: $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b}$.

EXAMPLE

$A = (3, 4)$, $B = (4, 5)$:
$\overrightarrow{AB} = (1, 1)$.
$A' = (-3, 2)$, $B' = (-2, 3)$:
$\overrightarrow{A'B'} = (1, 1)$.

YOUR TURN · one minute

Translate $(2, -1)$ by $(3, 2)$. Which vector leads back? **Answer:** $(5, 1)$; $(-3, -2)$.

Points and vectors are different objects

Both are written as pairs or triples of numbers. But:

$$
\begin{aligned}
\text{point} + \text{vector} &= \text{point} \\
\text{point} - \text{point} &= \text{vector} \\
\text{vector} \pm \text{vector} &= \text{vector} \\
\text{point} + \text{point} &= \text{no meaning}
\end{aligned}
$$

CAREFUL · moving the origin

changes the coordinates of every *point*, but of no *vector*.

Everything today uses only addition and scaling, so it works in any affine frame (oblique axes, different units). Lengths and angles, from Lecture 2, need a Cartesian frame.

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Segments, rays and lines

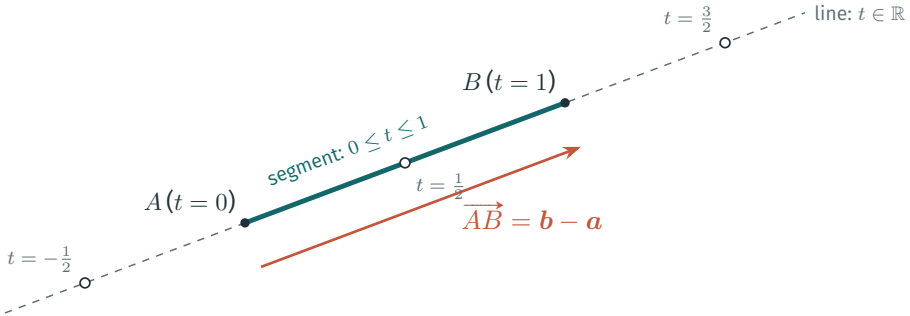
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A fraction of a journey

For $A \neq B$, travel from A a fraction t of the displacement $\overrightarrow{AB}$:

$$P(t) = A + t\,\overrightarrow{AB}, \qquad \boldsymbol{p}(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) = (1-t)\,\boldsymbol{a} + t\,\boldsymbol{b}.$$



Think of t as time: at $t = 0$ we are at A, at $t = 1$ at B, with uniform motion in between.

Segment, ray, line: the range is part of the answer

PROPOSITION

Let $A \neq B$ and $\boldsymbol{p}(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$. Distinct values of t give distinct points, and

$0 \leq t \leq 1$	the segment $[AB]$
$t \geq 0$	the ray from A through B
$t \in \mathbb{R}$	the line AB

Why the segment: $\overrightarrow{AP} = t\,\overrightarrow{AB}$ and $\overrightarrow{PB} = (1-t)\,\overrightarrow{AB}$ must both point from A towards B, so $t \geq 0$ and $1 - t \geq 0$.

Back to the moving marker

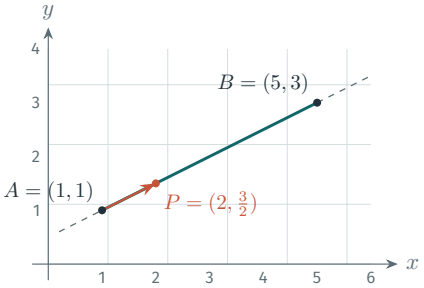
$A = (1, 1)$, $B = (5, 3)$.

1. Displacement: $\overrightarrow{AB} = (4, 2)$.
2. A quarter of it: $(1, \frac{1}{2})$.
3. Start from A:
   $P = (1, 1) + (1, \frac{1}{2}) = (2, \frac{3}{2})$.

CAREFUL

Not a quarter of the coordinates of B: we start from A, not from O.

$t = \frac{1}{2}$: the midpoint $(3, 2)$. $t = -\frac{1}{2}$: the point $(-1, 0)$, on the line, behind A.



Is the point on the line?

$A = (1, 2, 3)$, $B = (3, 0, 5)$. Are $C = (5, -2, 7)$ and $D = (2, 1, 5)$ on the line AB? On $[AB]$?

EXAMPLE · the point C

$\overrightarrow{AB} = (2, -2, 2)$ and $\overrightarrow{AC} = (4, -4, 4) = 2\,\overrightarrow{AB}$.

On the line, with $t = 2$; not on the segment (beyond B).

EXAMPLE · the point D

$\overrightarrow{AD} = (1, -1, 2)$. From $t(2, -2, 2) = (1, -1, 2)$: first coordinate $t = \frac{1}{2}$, third coordinate $t = 1$.

No such t: D is not on the line.

No division by a coordinate that might be zero, no special case for vertical lines: just compare with multiples of $\overrightarrow{AB}$.

Dividing a segment in a given ratio

PROPOSITION · division point

Let $m, n > 0$. The point $P \in [AB]$ with $AP : PB = m : n$ is

$$\boldsymbol{p} = \frac{n\,\boldsymbol{a} + m\,\boldsymbol{b}}{m+n} \quad \left(t = \tfrac{m}{m+n}\right); \qquad \text{midpoint: } \boldsymbol{m} = \tfrac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).$$

The larger weight goes to the *nearer* endpoint: $AP : PB = 1 : 3$ gives $\frac{1}{4}(3\boldsymbol{a} + \boldsymbol{b})$.

CAREFUL · if $A = B$

$\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) = \boldsymbol{a}$ for every t: a single point, not a line. A line needs a *nonzero* direction vector.

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Affine combinations

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Move the origin, not the points

New origin O' with $\overrightarrow{OO'} = \boldsymbol{o}$; the position vector of A becomes $\boldsymbol{a} - \boldsymbol{o}$.

EXAMPLE · midpoint

$$\tfrac{1}{2}(\boldsymbol{a} - \boldsymbol{o}) + \tfrac{1}{2}(\boldsymbol{b} - \boldsymbol{o}) = \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{o}$$

The *same point*, seen from O'.

CAREFUL · sum

$$(\boldsymbol{a} - \boldsymbol{o}) + (\boldsymbol{b} - \boldsymbol{o}) = (\boldsymbol{a} + \boldsymbol{b}) - 2\boldsymbol{o}$$

Not $(\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{o}$: "the point $\boldsymbol{a} + \boldsymbol{b}$" moves with the origin.

The difference: the coefficients of the midpoint formula add up to 1.

Affine combinations

PROPOSITION · origin independence

Let $A_1, \dots, A_k$ be points and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$.

- If $\sum \lambda_i = 1$, the point with position vector $\sum \lambda_i \boldsymbol{a}_i$ does not depend on the origin.
- If $\sum \lambda_i = 0$, the vector $\sum \lambda_i \boldsymbol{a}_i$ does not depend on the origin.

Proof in one line: $\sum \lambda_i(\boldsymbol{a}_i - \boldsymbol{o}) = \sum \lambda_i \boldsymbol{a}_i - \big(\sum \lambda_i\big)\boldsymbol{o}$.

Coefficient sum 1: a point (an *affine combination*; *convex* if all $\lambda_i \geq 0$).
Coefficient sum 0: a vector. Anything else: no geometric meaning.

Point, vector, or neither?

YOUR TURN · two minutes

Which of these describe a point independent of the origin, which a vector, and which neither?

$$\tfrac{1}{2}\boldsymbol{a}+\tfrac{1}{2}\boldsymbol{b}, \qquad \boldsymbol{a}+\boldsymbol{b}, \qquad 2\boldsymbol{a}-\boldsymbol{b}, \qquad \boldsymbol{a}-\boldsymbol{b}, \qquad \tfrac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}).$$

- $\frac{1}{2}\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}$: the midpoint of $[AB]$.
- $2\boldsymbol{a}-\boldsymbol{b}$: a point, the reflection of B in A.
- $\boldsymbol{a}-\boldsymbol{b}=\overrightarrow{BA}$: a vector. $\frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$: a point.
- $\boldsymbol{a}+\boldsymbol{b}$: neither.

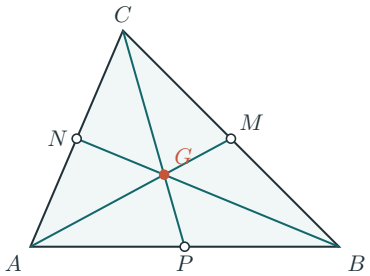
The centroid of a triangle

THEOREM · centroid

For non-collinear A, B, C, the three medians pass through

$$\boldsymbol{g} = \frac{\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}}{3},$$

which lies two thirds of the way from each vertex to the midpoint of the opposite side.



Proof of the centroid theorem

1. Let M be the midpoint of $[BC]$: $\boldsymbol{m} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$.
2. The point of $[AM]$ with ratio $2:1$ from A:

$$\frac{1 \cdot \boldsymbol{a} + 2\,\boldsymbol{m}}{3} = \frac{\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}}{3}.$$

3. The result is symmetric in $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$: the medians from B and from C give the *same* point.

APPLICATION · balance point

G balances three equal masses at the vertices. By origin independence it does not depend on the frame, as a physical balance point must.

For four points, $\frac{1}{4}(A + B + C + D)$ is in general *not* the balance point of the filled quadrilateral.

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Interpolation and Bézier curves

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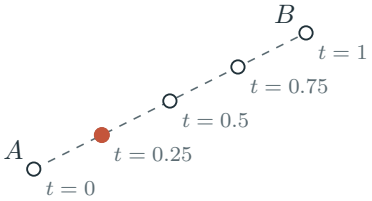
Linear interpolation

Animation: an object at A at time 0 and at B at time 1 is drawn at

$$(1 - t)\,A + t\,B$$

in every frame of the film.

- The weights add up to 1: the answer does not depend on the program's coordinate system.
- The same formula blends colours, sounds, camera positions…

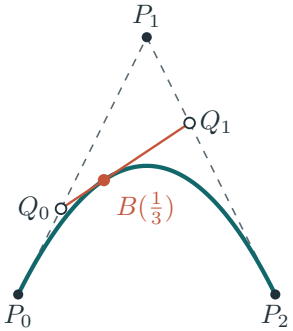


Smooth curves from repeated interpolation

Control points P_0, P_1, P_2; for $t \in [0,1]$:

$$\begin{aligned} Q_0 &= (1-t)P_0 + tP_1, \\ Q_1 &= (1-t)P_1 + tP_2, \\ B(t) &= (1-t)Q_0 + tQ_1 \\ &= (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2. \end{aligned}$$

The weights add up to $((1-t)+t)^2 = 1$ and are ≥ 0, so $B(t)$ stays in the triangle $P_0P_1P_2$. This is the *de Casteljau algorithm*; B traces a *quadratic Bézier curve*.



A Bézier arc

$P_0 = (0,0)$, $P_1 = (2,4)$, $P_2 = (4,0)$.

EXAMPLE · at $t = \frac{1}{3}$

$Q_0 = \frac{2}{3}P_0 + \frac{1}{3}P_1 = (\frac{2}{3}, \frac{4}{3})$,
$Q_1 = \frac{2}{3}P_1 + \frac{1}{3}P_2 = (\frac{8}{3}, \frac{8}{3})$,
$B(\frac{1}{3}) = \frac{2}{3}Q_0 + \frac{1}{3}Q_1 = (\frac{4}{3}, \frac{16}{9})$.

EXAMPLE · for every t

$B(t) = 2t(1-t)(2,4) + t^2(4,0) = (4t,\ 8t(1-t))$.

With $x = 4t$: the parabola $y = 2x - \frac{1}{2}x^2$, top at $B(\frac{1}{2}) = (2,2)$.

Where you meet them: outlines of letters in TrueType fonts, paths in SVG files and drawing programs, animation curves.

Summary

- A point is a position; a vector is a displacement. $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$, and $P + \boldsymbol{v}$ is a point.
- Chasles: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Scaling: length $\times|\lambda|$, sense by the sign of λ.
- Non-collinear $\boldsymbol{u}, \boldsymbol{v}$: the coefficients in $\alpha\boldsymbol{u} + \beta\boldsymbol{v}$ are unique.
- For $A \neq B$: $\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$ is the segment ($0 \leq t \leq 1$), the ray ($t \geq 0$), the line ($t \in \mathbb{R}$).
- Coefficient sum 1: a point; sum 0: a vector; neither depends on the origin.
- Midpoint $\frac{1}{2}(A + B)$, division point $\frac{nA+mB}{m+n}$, centroid $\frac{1}{3}(A + B + C)$.

Seminar 1: what to prepare

Prepare for the first meeting: Problem 1.1 (Core). You may be asked to present it at the board.

- Also accessible now: Problem 1.4 (Medium) and board problems 1.B1–1.B5.
- They are choices for discussion, not homework to finish.

NEXT LECTURE

Lengths, angles and orthogonal projection. Problems 1.2, 1.3, 1.5, 1.6, 1.7 and 1.B6–1.B8 use it.

The sheet opens with a one-page summary of the facts you need.