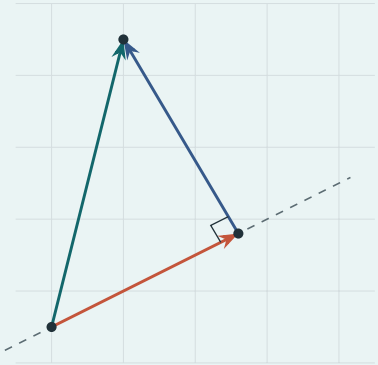
ANALYTIC GEOMETRY · LECTURE 2

Lengths, angles and orthogonal projection

How long? At what angle? How close?



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RECALL FROM LECTURE 1

Quick quiz: which direction?

YOUR TURN · 3 minutes — work individually

Let A, B, C be non-collinear points. Which vector equals

$$\overrightarrow{AB} - \overrightarrow{AC}\,?$$

A. $\overrightarrow{BC}$

C. $\overrightarrow{BA}$

B. $\overrightarrow{CB}$

D. $\overrightarrow{CA}$

ON A PIECE OF PAPER

Write **one letter (A–D)** and a **one-line justification**.

Three questions for today

YOUR TURN · distance

How far apart are $(1, 0, 2)$ and $(4, 5, 3)$? What does “far” mean in coordinates?

YOUR TURN · closest point

Which point of a given line is closest to a given point? Can we *prove* it, without calculus?

YOUR TURN · similarity

How can a search engine tell that two documents are about the same subject?

One operation answers all three: the *dot product*. From today on, every frame is *Cartesian*: perpendicular basic vectors of length 1.

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Length and distance

IN THIS LECTURE

1 Length and distance
2 The dot product
3 Orthogonal projection
4 The closest point of a line
5 Two inequalities
6 Distance or direction?

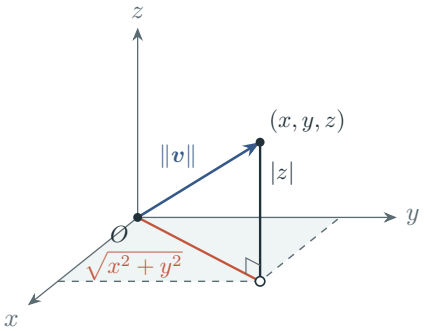
Length from Pythagoras

DEFINITION · norm, distance

$\|\boldsymbol{v}\| = \sqrt{x^2 + y^2 + z^2}$ for $\boldsymbol{v} = (x, y, z)$,

$d(A, B) = \left\|\overrightarrow{AB}\right\| = \|\boldsymbol{b} - \boldsymbol{a}\|$.

Pythagoras twice: the floor diagonal has length $\sqrt{x^2 + y^2}$, the vertical edge $|z|$, and they are perpendicular.



Unit vectors

PROPOSITION

$\|\boldsymbol{v}\| \geq 0$, with equality only for $\boldsymbol{v} = \mathbf{0}$;
$\|\lambda \boldsymbol{v}\| = |\lambda| \, \|\boldsymbol{v}\|$.

For $\boldsymbol{v} \neq \mathbf{0}$, the *unit vector* in its direction is $\widehat{\boldsymbol{v}} = \boldsymbol{v} / \|\boldsymbol{v}\|$. $\mathbf{0}$ cannot be normalised: it has no direction.

EXAMPLE

$P = (1, -2)$, $Q = (5, 1)$:
$\overrightarrow{PQ} = (4, 3)$, $d(P, Q) = 5$.

$\boldsymbol{u} = (2, -1, 2)$: $\|\boldsymbol{u}\| = 3$,
$\widehat{\boldsymbol{u}} = (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$.

CAREFUL

$d(P, Q)$ is not $\|\boldsymbol{q}\| - \|\boldsymbol{p}\|$. And $\sqrt{x^2 + y^2}$ is *not* a length in an oblique frame.

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The dot product

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The dot product

DEFINITION · dot product

$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$ — a *number*, not a vector.

PROPOSITION · algebraic rules

$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

$(\lambda \boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \lambda\, \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$

$\boldsymbol{u} \cdot \boldsymbol{u} = \|\boldsymbol{u}\|^2$

Expanding like $(a \pm b)^2$:

$$\|\boldsymbol{u} \pm \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \pm 2\, \boldsymbol{u} \cdot \boldsymbol{v} + \|\boldsymbol{v}\|^2 .$$

All three rules are one line of coordinates each.

What the dot product measures

THEOREM

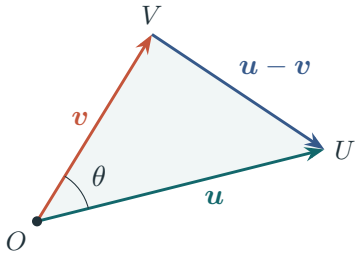
For nonzero $\boldsymbol{u}, \boldsymbol{v}$ with angle $\theta \in [0, \pi]$:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta.$$

Proof. Compute $UV^2 = \|\boldsymbol{u} - \boldsymbol{v}\|^2$ twice:

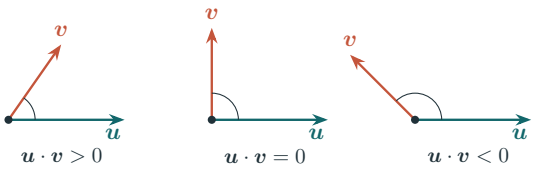
- law of cosines: $\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\,\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos\theta$;
- expansion: $\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\,\boldsymbol{u}\cdot\boldsymbol{v}$.

Collinear vectors ($\theta = 0$ or π): check directly.



Angles and perpendicular vectors

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \qquad (\boldsymbol{u},\boldsymbol{v}\neq\mathbf{0},\ \ 0\le\theta\le\pi)$$



Nonzero $\boldsymbol{u},\boldsymbol{v}$ are perpendicular exactly when $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

CAREFUL

$\mathbf{0}\cdot\boldsymbol{v}=0$ always, but $\mathbf{0}$ forms no angle with anything.

Angles from coordinates

EXAMPLE · in the plane

$\boldsymbol{a} = (1, 0)$, $\boldsymbol{b} = (1, \sqrt{3})$.

$\boldsymbol{a} \cdot \boldsymbol{b} = 1$, $\|\boldsymbol{a}\| = 1$, $\|\boldsymbol{b}\| = 2$,
$\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$.

EXAMPLE · a triangle in space

$A = (1, 0, 0)$, $B = (0, 1, 0)$,
$C = (0, 0, 1)$.
$\overrightarrow{AB} = (-1, 1, 0)$, $\overrightarrow{AC} = (-1, 0, 1)$:
dot product 1, norms $\sqrt{2}$, angle $\frac{\pi}{3}$.
By symmetry: *equilateral*.

YOUR TURN · one minute

Which angle of the triangle $A = (0, 0)$, $B = (4, 2)$, $C = (-1, 2)$ is right?
Answer: at A: $\overrightarrow{AB} \cdot \overrightarrow{AC} = -4 + 4 = 0$.

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Orthogonal projection

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Splitting a vector along a direction

PROPOSITION · orthogonal projection

Let $\boldsymbol{v} \neq \mathbf{0}$. There is exactly one t with $(\boldsymbol{u} - t\boldsymbol{v}) \perp \boldsymbol{v}$:

$$t_* = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}. \qquad \operatorname{proj}_{\boldsymbol{v}} \boldsymbol{u} = t_* \boldsymbol{v}, \qquad \boldsymbol{r} = \boldsymbol{u} - t_* \boldsymbol{v} \perp \boldsymbol{v}.$$

Proof. $(\boldsymbol{u} - t\boldsymbol{v}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} - t\, \boldsymbol{v} \cdot \boldsymbol{v}$, and $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2 > 0$.

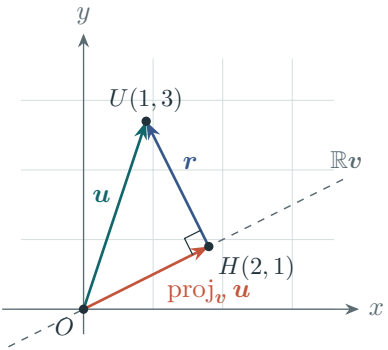
$\boldsymbol{u} = \operatorname{proj}_{\boldsymbol{v}} \boldsymbol{u} + \boldsymbol{r}$: a part along $\boldsymbol{v}$ plus a part perpendicular to $\boldsymbol{v}$ — the shadow of $\boldsymbol{u}$ at noon.

Example: a projection

Project $\boldsymbol{u} = (1, 3)$ onto $\boldsymbol{v} = (2, 1)$.

1. $\boldsymbol{u} \cdot \boldsymbol{v} = 5$, $\boldsymbol{v} \cdot \boldsymbol{v} = 5$, so $t_* = 1$.
2. $\operatorname{proj}_{\boldsymbol{v}} \boldsymbol{u} = (2, 1)$.
3. $\boldsymbol{r} = (1, 3) - (2, 1) = (-1, 2)$.
4. Check: $\boldsymbol{r} \cdot \boldsymbol{v} = -2 + 2 = 0$.

The distance from U to the line $\mathbb{R}\boldsymbol{v}$ is $\|\boldsymbol{r}\| = \sqrt{5}$.



A number, a vector and a length

Three different objects:

- the coefficient t_* (a number);
- the projection $t_*\boldsymbol{v}$ (a vector);
- the signed length of the shadow, $\boldsymbol{u} \cdot \widehat{\boldsymbol{v}} = t_* \, \|\boldsymbol{v}\|$.

PROPOSITION · only the line matters

Replace $\boldsymbol{v}$ by $c\boldsymbol{v}$, $c \neq 0$: the coefficient becomes t_*/c, the projection stays $t_*\boldsymbol{v}$.

YOUR TURN

Project (a, b) onto $(0, 1)$. Do you need a slope? **Answer:** $(0, b)$, remainder $(a, 0)$. No slope, no division by a coordinate.

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The closest point of a line

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Why the foot is the closest point

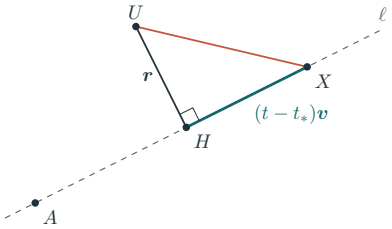
THEOREM · closest point

Line ℓ through A with direction $\boldsymbol{v} \neq \mathbf{0}$; a point U. Put

$$t_* = \frac{\overrightarrow{AU} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}, \quad H = A + t_* \boldsymbol{v}, \quad \boldsymbol{r} = \overrightarrow{HU}.$$

For every $X = A + t\boldsymbol{v}$ on ℓ:

$$UX^2 = \|\boldsymbol{r}\|^2 + (t - t_*)^2 \, \|\boldsymbol{v}\|^2 .$$



So H is the *unique* closest point, $UH \perp \ell$, and $d(U, \ell) = \|\boldsymbol{r}\|$.

Proof: Pythagoras for every point of the line

1. $\boldsymbol{r} = \overrightarrow{AU} - t_*\boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$ (projection).
2. For $X = A + t\boldsymbol{v}$: $\overrightarrow{XU} = \overrightarrow{AU} - t\boldsymbol{v} = \boldsymbol{r} + (t_* - t)\,\boldsymbol{v}$.
3. Expand:
$$UX^2 = \|\boldsymbol{r}\|^2 + 2(t_* - t)\underbrace{\boldsymbol{r}\cdot\boldsymbol{v}}_{=0} + (t_* - t)^2\,\|\boldsymbol{v}\|^2 .$$
4. The last term is ≥ 0, and $= 0$ only for $t = t_*$, because $\|\boldsymbol{v}\|^2 > 0$.

CAREFUL

Trying a few values of t proves nothing. The identity holds for *all* t at once.

Example: distance from a point to a line

EXAMPLE · in the plane

$\ell:\ (0,1)+t(2,1),\ U=(1,4).$
$\overrightarrow{AU}=(1,3)$ — not the position vector of U!
$t_*=\frac{5}{5}=1,\ H=(2,2),$
$\boldsymbol{r}=(-1,2),\ d(U,\ell)=\sqrt{5}.$

EXAMPLE · in space

$\ell:\ (1,0,2)+t(1,2,2),$
$V=(4,5,3).$
$\overrightarrow{AV}=(3,5,1),\ t_*=\frac{15}{9}=\frac{5}{3},$
$H=(\frac{8}{3},\frac{10}{3},\frac{16}{3}),\ \overrightarrow{HV}=(\frac{4}{3},\frac{5}{3},-\frac{7}{3}),$
check $\overrightarrow{HV}\cdot\boldsymbol{v}=0;\ d(V,\ell)=\sqrt{10}.$

First two questions of the day: the points $(1,0,2)$ and $(4,5,3)$ are $\sqrt{9+25+1}=\sqrt{35}$ apart, and the closest point of a line is the foot of the perpendicular.

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Two inequalities

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The Cauchy–Schwarz inequality

THEOREM · Cauchy–Schwarz

$|\boldsymbol{u}\cdot\boldsymbol{v}| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$, with equality exactly when $\boldsymbol{u},\boldsymbol{v}$ are collinear.

Proof. For $\boldsymbol{v}\neq\mathbf{0}$, take $A=O$, $\overrightarrow{OU}=\boldsymbol{u}$ and $t=0$ in the closest-point identity:

$$\|\boldsymbol{u}\|^2 = \|\boldsymbol{r}\|^2 + t_*^2\,\|\boldsymbol{v}\|^2 \;\ge\; \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{\|\boldsymbol{v}\|^2}.$$

Equality exactly when $\boldsymbol{r}=\mathbf{0}$, i.e. $\boldsymbol{u}=t_*\boldsymbol{v}$.

The proof uses only the algebraic rules, so it works for vectors with any number of coordinates.

The triangle inequality, and an optimisation

PROPOSITION · triangle inequality

$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, so $d(A,C) \le d(A,B) + d(B,C)$.

Proof: $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2\,\boldsymbol{u}\cdot\boldsymbol{v} + \|\boldsymbol{v}\|^2 \le (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2$ by Cauchy–Schwarz.

EXAMPLE · no calculus needed

Largest value of $2x + 3y + 6z$ when $x^2 + y^2 + z^2 = 1$?

It is $(2,3,6)\cdot(x,y,z) \le \sqrt{4+9+36}\cdot 1 = 7$, attained only at $\frac{1}{7}(2,3,6)$.

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Distance or direction?

IN THIS LECTURE

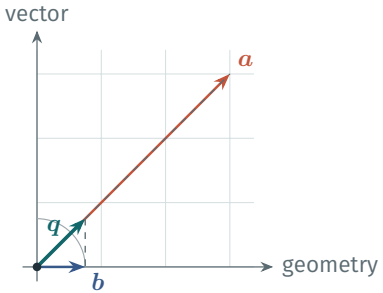
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Documents as vectors

Counts of (*geometry*, *vector*): query $\boldsymbol{q} = (1, 1)$, documents $\boldsymbol{a} = (4, 4)$, $\boldsymbol{b} = (1, 0)$.

- Distance: $\|\boldsymbol{a} - \boldsymbol{q}\| = 3\sqrt{2} \approx 4.24$, $\|\boldsymbol{b} - \boldsymbol{q}\| = 1$: prefers $\boldsymbol{b}$.
- Direction: $\operatorname{cosim}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|}$; $\operatorname{cosim}(\boldsymbol{q}, \boldsymbol{a}) = 1$, $\operatorname{cosim}(\boldsymbol{q}, \boldsymbol{b}) = \frac{1}{\sqrt{2}} \approx 0.71$: prefers $\boldsymbol{a}$.

Real systems use one coordinate per word, thousands in all; Cauchy–Schwarz still gives $\operatorname{cosim} \in [-1, 1]$.



Normalise, then compare

PROPOSITION

For unit vectors: $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2 - 2\,\boldsymbol{x} \cdot \boldsymbol{y}$.

- After normalising, ranking by distance and ranking by cosine similarity agree.
- Example: $\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{q}}$ (distance 0), $\left\|\widehat{\boldsymbol{b}} - \widehat{\boldsymbol{q}}\right\| = \sqrt{2 - \sqrt{2}} \approx 0.77$.

WHAT NORMALISING DISCARDS

The length of the vector. For “what is this document about?” that is usually right; for “which document is longer, or repeats itself?” it is not. The mathematics does not choose the question for you.

Summary

- Cartesian frame: $\|\boldsymbol{v}\| = \sqrt{x^2 + y^2 + z^2}$, $d(A, B) = \|\boldsymbol{b} - \boldsymbol{a}\|$, $\widehat{\boldsymbol{v}} = \boldsymbol{v} / \|\boldsymbol{v}\|$.
- $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta$; nonzero $\boldsymbol{u} \perp \boldsymbol{v} \iff \boldsymbol{u} \cdot \boldsymbol{v} = 0$.
- Projection onto $\boldsymbol{v} \neq \boldsymbol{0}$: $t_* = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}$, $\operatorname{proj}_{\boldsymbol{v}} \boldsymbol{u} = t_* \boldsymbol{v}$, $\boldsymbol{u} - t_* \boldsymbol{v} \perp \boldsymbol{v}$.
- Closest point: $UX^2 = UH^2 + (t - t_*)^2 \, \|\boldsymbol{v}\|^2$.
- Cauchy–Schwarz and the triangle inequality.
- Distance compares positions; cosine similarity compares directions.

WRAP-UP

Seminar 1, second meeting: what to prepare

Prepare: Problems 1.2 and 1.3 (Core). You may be asked to present them at the board.

- Also for this meeting: Problems 1.5, 1.6 (Medium), 1.7 (Challenge), and board problems 1.B6–1.B8.
- They are choices for discussion, not homework to finish.

NEXT LECTURE

Lines: equations, intersections and constraints — in the plane and in space.