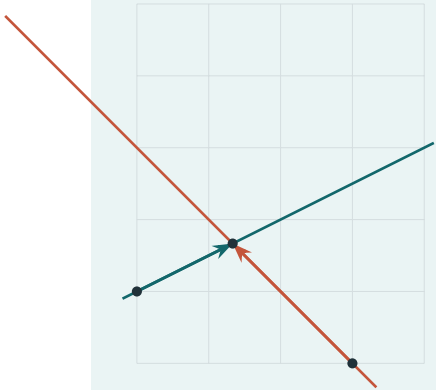
ANALYTIC GEOMETRY · LECTURE 3

Lines, intersections and constraints

When do two straight paths really meet?



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RECALL FROM LECTURE 2

Quick quiz: which vector do we project?

YOUR TURN · 2 minutes — work individually

ℓ is the line through $A = (0, 1)$ with direction $\boldsymbol{v} = (2, 1)$, and $U = (1, 4)$. To find the point of ℓ closest to U, we project a vector onto $\boldsymbol{v}$. Which one?

A. $\boldsymbol{u} = (1, 4)$, the position vector of U

B. $\overrightarrow{AU} = (1, 3)$

C. $\boldsymbol{v} = (2, 1)$ itself

D. $\overrightarrow{UA} = (-1, -3)$

ON A PIECE OF PAPER

Write **one letter (A–D)** and a **one-line justification**.

Three questions for today

YOUR TURN · closest point

Why is the foot of the perpendicular closer to U than *every* other point of the line? Can we prove it for all points at once?

YOUR TURN · one line, many names

$(1, 2) + t(2, -1)$ and $(5, 0) + s(-4, 2)$ look different. Are they the same line? How do we decide quickly?

YOUR TURN · crossing paths

Two straight paths cross on the map. Do the two walkers actually meet?

We first finish what we started in Lecture 2, then turn to lines: how to describe one, and how two of them can sit relative to each other.

1

The closest point of a line (from Lecture 2)

IN THIS LECTURE

1 The closest point of a line (from Lecture 2)
2 Two inequalities
3 Distance or direction?
4 Describing a line
5 When two lines meet
6 Rays, segments and moving points

Why the foot is the closest point

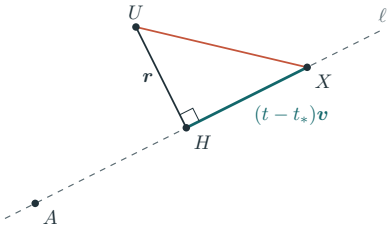
THEOREM · closest point

Line ℓ through A with direction $\boldsymbol{v} \neq \mathbf{0}$; a point U. Put

$$t_* = \frac{\overrightarrow{AU} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}, \quad H = A + t_* \boldsymbol{v}, \quad \boldsymbol{r} = \overrightarrow{HU}.$$

For every $X = A + t\boldsymbol{v}$ on ℓ:

$$UX^2 = \|\boldsymbol{r}\|^2 + (t - t_*)^2 \, \|\boldsymbol{v}\|^2 .$$



So H is the *unique* closest point, $UH \perp \ell$, and $d(U, \ell) = \|\boldsymbol{r}\|$.

Proof: Pythagoras for every point of the line

1. $\boldsymbol{r} = \overrightarrow{AU} - t_*\boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$: that is what projection means.
2. For $X = A + t\boldsymbol{v}$: $\overrightarrow{XU} = \overrightarrow{AU} - t\boldsymbol{v} = \boldsymbol{r} + (t_* - t)\,\boldsymbol{v}$.
3. Expand the square:

$$UX^2 = \|\boldsymbol{r}\|^2 + 2(t_* - t)\underbrace{\boldsymbol{r}\cdot\boldsymbol{v}}_{=0} + (t_* - t)^2\,\|\boldsymbol{v}\|^2 .$$

4. The last term is ≥ 0, and $= 0$ only for $t = t_*$, because $\|\boldsymbol{v}\|^2 > 0$. $\square$

CAREFUL

Trying a few values of t proves nothing. The identity holds for *all* t at once — that is why it is a proof.

Example: distance from a point to a line

EXAMPLE · the quiz, in the plane

$\ell:\ (0,1)+t(2,1),\ U=(1,4).$
$\overrightarrow{AU}=(1,3)$ — answer **B**.
$t_*=\frac{2+3}{5}=1,\ H=(2,2),$
$\boldsymbol{r}=(-1,2),$
$d(U,\ell)=\sqrt{5}$; indeed
$UX^2=5+5(t-1)^2.$

EXAMPLE · in space

$\ell:\ (1,0,2)+t(1,2,2),$
$V=(4,5,3).$
$\overrightarrow{AV}=(3,5,1),\ t_*=\frac{15}{9}=\frac{5}{3},$
$H=(\frac{8}{3},\frac{10}{3},\frac{16}{3}),\ \overrightarrow{HV}=(\frac{4}{3},\frac{5}{3},-\frac{7}{3}),$
check $\overrightarrow{HV}\cdot\boldsymbol{v}=0$; $d(V,\ell)=\sqrt{10}.$

Project the displacement from a point *of the line*; it equals the position vector of U only when $A=O$.

2

Two inequalities

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The Cauchy–Schwarz inequality

THEOREM · Cauchy–Schwarz

$|\boldsymbol{u}\cdot\boldsymbol{v}| \le \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$, with equality exactly when $\boldsymbol{u},\boldsymbol{v}$ are collinear.

Proof. For $\boldsymbol{v}\neq\boldsymbol{0}$, take $A=O$, $\overrightarrow{OU}=\boldsymbol{u}$ and $t=0$ in the closest-point identity:

$$\|\boldsymbol{u}\|^2 = \|\boldsymbol{r}\|^2 + t_*^2\,\|\boldsymbol{v}\|^2 \;\ge\; t_*^2\,\|\boldsymbol{v}\|^2 = \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{\|\boldsymbol{v}\|^2}.$$

Equality exactly when $\boldsymbol{r}=\boldsymbol{0}$, i.e. $\boldsymbol{u}=t_*\boldsymbol{v}$. (For $\boldsymbol{v}=\boldsymbol{0}$ both sides are 0.) □

The proof uses only the algebraic rules of the dot product, so it works for vectors with any number of coordinates.

The triangle inequality, and an optimisation

PROPOSITION · triangle inequality

$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, so $d(A,C) \le d(A,B) + d(B,C)$.

Proof: expand $\|\boldsymbol{u}+\boldsymbol{v}\|^2$ and bound $2\,\boldsymbol{u}\cdot\boldsymbol{v}$ by Cauchy–Schwarz. Equality for nonzero vectors: same direction *and* sense, i.e. $B \in [AC]$.

EXAMPLE · no calculus needed

Largest value of $2x+3y+6z$ when $x^2+y^2+z^2=1$?

It is $(2,3,6)\cdot(x,y,z) \le \sqrt{4+9+36}\cdot 1 = 7$, attained only at $\frac{1}{7}(2,3,6)$.

A bound is not a maximum until it is attained; the equality case says where.

3

Distance or direction?

IN THIS LECTURE

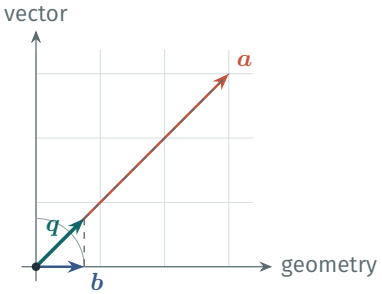
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Documents as vectors

Counts of (*geometry*, *vector*): query $\boldsymbol{q} = (1, 1)$, documents $\boldsymbol{a} = (4, 4)$, $\boldsymbol{b} = (1, 0)$.

- Distance: $\|\boldsymbol{a} - \boldsymbol{q}\| = 3\sqrt{2} \approx 4.24$, $\|\boldsymbol{b} - \boldsymbol{q}\| = 1$: prefers $\boldsymbol{b}$.
- Direction: $\operatorname{cosim}(\boldsymbol{x}, \boldsymbol{y}) = \dfrac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|}$; $\operatorname{cosim}(\boldsymbol{q}, \boldsymbol{a}) = 1$, $\operatorname{cosim}(\boldsymbol{q}, \boldsymbol{b}) = \frac{1}{\sqrt{2}} \approx 0.71$: prefers $\boldsymbol{a}$.

Real systems use one coordinate per word, thousands in all; Cauchy–Schwarz still guarantees $\operatorname{cosim} \in [-1, 1]$.



Normalise, then compare

PROPOSITION

For unit vectors: $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2 - 2\,\boldsymbol{x} \cdot \boldsymbol{y}$.

- After normalising, ranking by distance and ranking by cosine similarity agree.
- Example: $\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{q}}$ (distance 0), $\left\|\widehat{\boldsymbol{b}} - \widehat{\boldsymbol{q}}\right\| = \sqrt{2 - \sqrt{2}} \approx 0.77$.

WHAT NORMALISING DISCARDS

The length of the vector. For “what is this document about?” that is usually right; for “which document is longer, or repeats itself?” it is not. The mathematics does not choose the question for you.

4

Describing a line

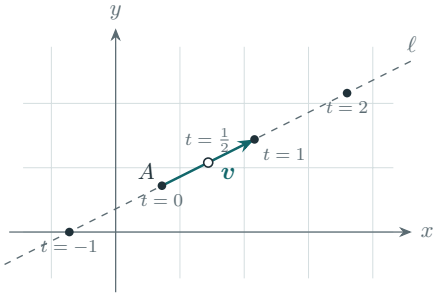
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A point, a direction, every real number

DEFINITION · line

$\ell = \{A + t\boldsymbol{v} : \ t \in \mathbb{R}\}$, with $\boldsymbol{v} \neq \mathbf{0}$.
A is a point, $\boldsymbol{v}$ a displacement, t a number.



In coordinates, with $A = (a_1, a_2, a_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$:

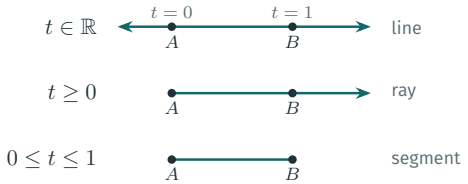
$$x = a_1 + tv_1, \quad y = a_2 + tv_2, \quad z = a_3 + tv_3,$$

the *same* t in every coordinate.
$\boldsymbol{v} = \mathbf{0}$ would give the single point A, not a line; and t is not a distance unless $\|\boldsymbol{v}\| = 1$.

Line, ray, segment: one formula, three domains

For distinct points A, B, with $\boldsymbol{v} = \overrightarrow{AB}$:

$$X(t) = A + t\,\overrightarrow{AB}.$$



- $t \in \mathbb{R}$: the line AB;
- $t \geq 0$: the ray from A through B;
- $0 \leq t \leq 1$: the segment $[AB]$.

YOUR TURN · half a minute

Where are $X(0)$, $X(1)$, $X(\frac{1}{2})$, $X(-1)$? **Answer:** A, B, the midpoint of $[AB]$, the reflection of B in A.

Is the point on the line?

$\ell:\ (x, y, z) = (1, -1, 2) + t(2, 1, 0)$, i.e. $x = 1 + 2t,\ y = -1 + t,\ z = 2$.

YOUR TURN · two minutes, in pairs

Test $P = (5, 1, 2)$ and $Q = (5, 1, 3)$. What is different about them?

EXAMPLE · P

$5 = 1 + 2t$ and $1 = -1 + t$ both give $t = 2$; and $2 = 2$. So $P \in \ell$.

EXAMPLE · Q

Again $t = 2$ from x, y — but the third equation says $3 = 2$. So $Q \notin \ell$.

A zero coordinate in the direction means a constant coordinate along the line ($z = 2$): keep that equation, never divide by 0.

Many descriptions, one line

EXAMPLE

$\ell:\ (1,2)+t(2,-1)$ and $m:\ (5,0)+s(-4,2)$. Same line?

- $(5,0)=(1,2)+2(2,-1)$ is on ℓ;
- $(-4,2)=-2\,(2,-1)$ is a nonzero multiple of the direction.

Hence $(5,0)+s(-4,2)=(1,2)+(2-2s)(2,-1)$, and as s runs through $\mathbb{R}$ so does $t=2-2s$: the same line.

PROPOSITION · same line

$B+s\boldsymbol{w}$ ($s\in\mathbb{R}$) describes the line $A+t\boldsymbol{v}$ exactly when $B\in\ell$ and $\boldsymbol{w}=c\boldsymbol{v}$ for some $c\neq 0$.

The rays $t\geq 0$ and $s\geq 0$ are different: other start, opposite sense.

Lines in the plane: an equation without slopes

$A = (a, b)$, $\boldsymbol{v} = (p, q) \neq (0, 0)$. From $x = a + tp$ and $y = b + tq$, eliminate t:

$$\boxed{q\,(x - a) - p\,(y - b) = 0.}$$

EXAMPLE · $p \neq 0$

Divide by p: $y - b = \frac{q}{p}\,(x - a)$, the point–slope form.

EXAMPLE · $p = 0$, so $q \neq 0$

$q(x - a) = 0$, i.e. $x = a$: a vertical line. No slope, no problem.

The boxed form covers every line, vertical ones included: it never divides by a coordinate of the direction.

The normal vector of a line in the plane

DEFINITION · normal equation

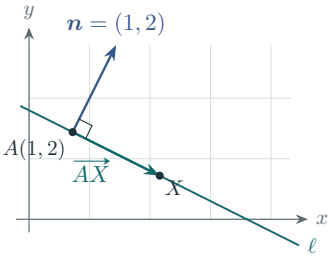
For a point $A = (a, b)$ and $\boldsymbol{n} = (\alpha, \beta) \neq \mathbf{0}$:

$$\boldsymbol{n} \cdot \overrightarrow{AX} = 0 \quad \Longleftrightarrow \quad \alpha x + \beta y = \alpha a + \beta b.$$

A direction of this line is $(-\beta, \alpha)$.

EXAMPLE

Through $(1, 2)$ with direction $(2, -1)$: take $\boldsymbol{n} = (1, 2)$.
$(x - 1) + 2(y - 2) = 0$, i.e. $x + 2y = 5$.



In space, one equation is not enough

A line in space needs *two* equations. If all $v_i \neq 0$, eliminating t gives the symmetric form

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}.$$

CAREFUL

For $(1, -1, 2) + t(2, 1, 0)$ this would divide by 0. Write $\frac{x-1}{2} = y + 1$ *and* $z = 2$ instead.

LOOKING AHEAD

One equation $\alpha x + \beta y + \gamma z = \delta$ with $(\alpha, \beta, \gamma) \neq \mathbf{0}$ is a *plane* (Lecture 4).

When in doubt, use the parametric form: it works the same way in the plane and in space, and never divides by anything.

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When two lines meet

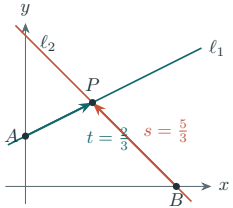
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Two lines, two parameters

$\ell_1 : \ A + t\boldsymbol{u}$ and $\ell_2 : \ B + s\boldsymbol{v}$ share a point exactly when

$$A + t\boldsymbol{u} = B + s\boldsymbol{v} \quad \text{for some } t, s \in \mathbb{R}.$$



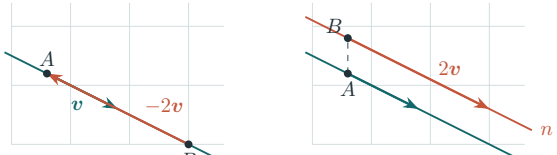
EXAMPLE

$\ell_1 : \ (0,1) + t(2,1), \ \ell_2 : \ (3,0) + s(-1,1).$
$2t = 3 - s$, $1 + t = s$: $t = \frac{2}{3}$, $s = \frac{5}{3}$,
$P = (\frac{4}{3}, \frac{5}{3})$.

Careful: each line labels its points with its own parameter. Insisting on $t = s$ asks a different question — here, one with no solution.

Parallel directions: the same line, or not?

$\ell:\ (1,2)+t(2,-1)$. Proportional directions ($\boldsymbol{v}=c\boldsymbol{u}$, $c\neq 0$): parallel lines; test one base point.



$\overrightarrow{AB}=(4,-2)=2\boldsymbol{v}$ $\overrightarrow{AB}=(0,1)$: not a multiple of $\boldsymbol{v}$

EXAMPLE

$m:\ (5,0)+s(-4,2)$: $B\in\ell$, so $m=\ell$.

EXAMPLE

$n:\ (1,3)+s(4,-2)$: $B\notin\ell$: distinct parallel lines.

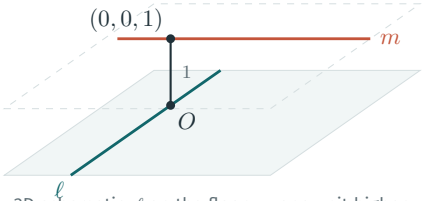
In space, the third coordinate can say no

$\ell:\ (0,0,0)+t(1,0,0)$,
$m:\ (0,1,1)+s(0,1,0)$.

- x: $t=0$; y: $0=1+s$, so $s=-1$;
- z: $0=1$. Impossible.

DEFINITION · skew lines

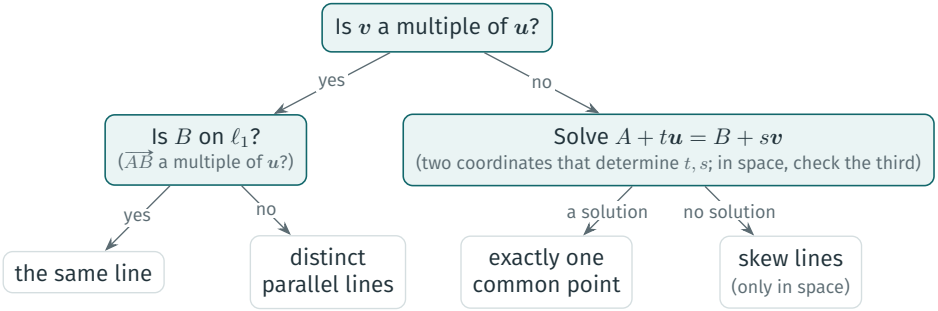
Lines in space that are not parallel and do not meet. They lie in no common plane.



3D schematic: ℓ on the floor, m one unit higher

Move the base point of m to $(0,1,0)$ and the same $t=0$, $s=-1$ now satisfy all three equations: the lines meet at O.

How two lines can sit: a procedure



Non-parallel lines share at most one point: two common points would give both lines the same direction. In the plane they always share one (notes); in space “no solution” means skew.

The angle between two lines

DEFINITION · angle between lines

For lines with directions $\boldsymbol{u}, \boldsymbol{v}$:

$$\cos\varphi = \frac{|\boldsymbol{u}\cdot\boldsymbol{v}|}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}, \quad 0 \le \varphi \le \tfrac{\pi}{2}.$$

A line has no preferred sense ($\boldsymbol{u}$ or $-\boldsymbol{u}$); the absolute value makes the formula agree. Only directions enter: skew lines have an angle too.

EXAMPLE

Directions $(2,-1)$ and $(1,2)$: dot product 0, so $\varphi = \frac{\pi}{2}$.

CAREFUL

The skew lines ℓ, m also have orthogonal directions, yet never meet. *Perpendicular lines*: orthogonal directions *and* a common point.

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Rays, segments and moving points

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Do the rays reach the crossing?

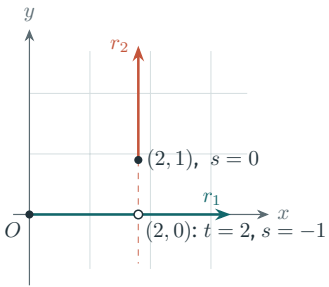
YOUR TURN · one minute

$r_1:\ (0,0)+t(1,0),\ t\geq 0,$ and $r_2:\ (2,1)+s(0,1),\ s\geq 0.$ Do the rays meet?

EXAMPLE

Lines: $t=2$, $s=-1$, point $(2,0)$. But $s<0$: it lies *behind* the start of r_2. **The rays do not meet.**

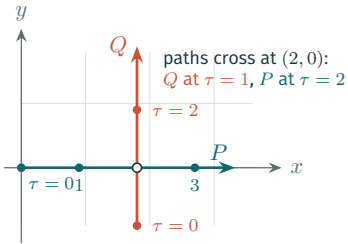
With direction $(0,-1)$ for r_2: $s=1$, and the rays meet at $(2,0)$.



Crossing paths, or a collision?

Two walkers, one clock $\tau \geq 0$: $P(\tau) = (\tau, 0)$, $Q(\tau) = (2, -1 + \tau)$.

- **Paths:** $P(t) = Q(s)$ gives $t = 2$, $s = 1$. They cross at $(2, 0)$.
- **Collision:** $P(\tau) = Q(\tau)$ needs $\tau = 2$ *and* $\tau = 1$. None: Q passes one time unit earlier.



Start Q at $(2, -2)$ instead: both reach $(2, 0)$ at $\tau = 2$ — a collision.
Rule: independent parameters for a geometric question; one shared parameter only when the clock is shared.

Exit check: explain, do not just calculate

YOUR TURN · two minutes, one sentence each

1. A line in space has direction $(0, 1, 0)$. Which coordinates are constant along it?
2. Two lines in space have perpendicular directions. What else must you check before calling the lines perpendicular?
3. Intersecting two lines gives $t = -2$ and $s = 3$. Is the point on both rays $t \geq 0$, $s \geq 0$?
4. $(0, 0) + t(1, 1)$ and $(2, 2) + s(-2, -2)$: the same line?

Answers: x and z; that the lines actually meet; no, $t = -2$ is forbidden; yes — $(2, 2)$ is on the first line and the directions are proportional, $t = 2 - 2s$.

Summary

- Closest point of $A + t\boldsymbol{v}$ to U: project $\overrightarrow{AU}$; $UX^2 = UH^2 + (t - t_*)^2 \, \|\boldsymbol{v}\|^2$ for every X.
- Cauchy–Schwarz $|\boldsymbol{u} \cdot \boldsymbol{v}| \le \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and the triangle inequality, with their equality cases; cosine similarity compares directions.
- A line: a point, a nonzero direction, and the domain $\mathbb{R}$ ($t \ge 0$ a ray, $0 \le t \le 1$ a segment). The same t in every coordinate.
- In the plane: $q(x - a) - p(y - b) = 0$, or $\boldsymbol{n} \cdot \overrightarrow{AX} = 0$. No slopes needed.
- Two lines: proportional directions $\to$ test a base point; otherwise solve with *independent* parameters and check every coordinate (skew lines in space).
- Rays and moving points: check the parameter domains, and whether the clock is shared.

WRAP-UP

Seminar 2: one sheet for two lectures

Prepare: every part of Problem 2.1 (Core). You may be asked to present it at the board.

- Also for this meeting: Problem 2.4 (Medium) and the board problems 2.B1–2.B5 (closest points, the two inequalities, cosine similarity, lines) — choices for discussion, not homework to finish.

AFTER LECTURE 4

Problems 2.2 and 2.3 (Core), 2.5 and 2.6 (Medium), 2.B6–2.B8; the Challenge 2.7 uses both lectures.

NEXT LECTURE

Planes: a normal vector, the cross product, and whether a ray hits a triangular face.