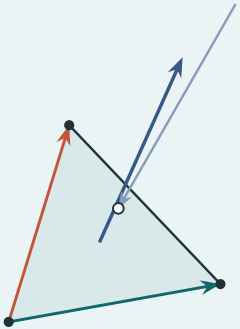
ANALYTIC GEOMETRY · LECTURE 4

Planes, normals, area and volume

Which way does a face point, and does the ray hit it?



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Babeș–Bolyai University · Mathematics and Computer Science · 2026–2027

Three questions for today

YOUR TURN · a face of a 3D model

A light ray reaches a triangular face of a 3D model. Which way does the face point, and does the ray hit it?

YOUR TURN · three points

Three points A, B, C in space, not on a line: what is the equation of the plane through them?

YOUR TURN · how much space

Four points in space span a tetrahedron. What is its volume, without a formula book?

One new operation answers all three: the *cross product*, which turns two edge vectors into a vector perpendicular to both. Coordinates today: a right-handed Cartesian frame with a common unit.

Quick check

YOUR TURN · two minutes, then compare with your neighbour

$A = (1, 0, 0)$, $B = (1, 2, 0)$, $C = (0, 0, 2)$ and $\boldsymbol{n} = (2, 0, 1)$.

1. Write $\overrightarrow{AB}$, $\overrightarrow{AC}$ and the line AB with its parameter domain.
2. Is $\boldsymbol{n}$ perpendicular to $\overrightarrow{AB}$? To $\overrightarrow{AC}$?
3. (A preview.) The plane through A perpendicular to $\boldsymbol{n}$: does it contain B? C? O?

EXAMPLE · answers

$(0, 2, 0)$, $(-1, 0, 2)$, $A + t(0, 2, 0)$ with $t \in \mathbb{R}$. Both dot products are 0.
B, C: yes, by the dot products. O: $\boldsymbol{n} \cdot \overrightarrow{AO} = (2, 0, 1) \cdot (-1, 0, 0) = -2 \neq 0$, no.

1

Two edges, one normal

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A vector perpendicular to two others

DEFINITION · cross product

In a right-handed Cartesian frame, for $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$:

$$\boldsymbol{u} \times \boldsymbol{v} = \big(u_2 v_3 - u_3 v_2,\ \ u_3 v_1 - u_1 v_3,\ \ u_1 v_2 - u_2 v_1\big).$$

A vector, not a number. Coordinate i skips index i; the pattern is cyclic, $1 \to 2 \to 3 \to 1$.

EXAMPLE

$(0, 2, 0) \times (-1, 0, 2) =$
$(4 - 0,\ 0 - 0,\ 0 + 2) = (4, 0, 2).$

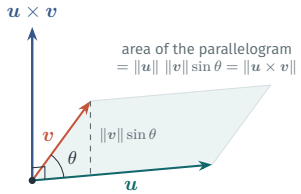
YOUR TURN · one minute

Is $(4, 0, 2) \perp$ both inputs? And $(-1, 0, 2) \times (0, 2, 0)$? **Answer:** yes; $(-4, 0, -2)$.

What the cross product measures

PROPOSITION

- $\boldsymbol{u} \times \boldsymbol{v} \perp \boldsymbol{u}$ and $\boldsymbol{u} \times \boldsymbol{v} \perp \boldsymbol{v}$;
- $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\| \sin\theta$: the area of the parallelogram spanned by $\boldsymbol{u}, \boldsymbol{v}$;
- $\boldsymbol{v} \times \boldsymbol{u} = -\,\boldsymbol{u} \times \boldsymbol{v}$; $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ exactly when $\boldsymbol{u}, \boldsymbol{v}$ are collinear;
- $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u} \times \boldsymbol{v})$ is right-handed, like $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$: $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$.



3D schematic; angles are not to scale

The length: expanding, $\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2$, and $1 - \cos^2\theta = \sin^2\theta$ (notes). Also $\times$ distributes over $+$.

One triangle for the whole lecture

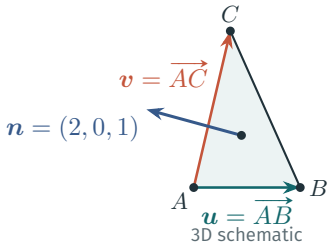
$A = (1, 0, 0)$, $B = (1, 2, 0)$, $C = (0, 0, 2)$:
$\overrightarrow{AB} \times \overrightarrow{AC} = (4, 0, 2)$.

YOUR TURN · two minutes

A simpler normal to the triangle? Its area? Are A, B, C collinear?

EXAMPLE · answers

$\boldsymbol{n} = (2, 0, 1)$ (any nonzero multiple is a normal). Area $= \frac{1}{2}\,\|(4, 0, 2)\| = \sqrt{5}$, half the parallelogram. Cross product $\neq \mathbf{0}$: not collinear.



3D schematic

Careful: rescale a normal freely, but take the area from the cross product of the edges.

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The equation of a plane

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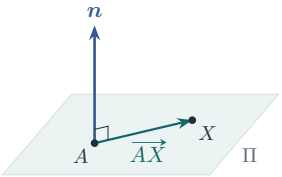
A plane: a point and a normal

DEFINITION · normal equation of a plane

The plane through A with normal $\boldsymbol{n} = (a, b, c) \neq \mathbf{0}$ is $\boldsymbol{n} \cdot \overrightarrow{AX} = 0$, i.e. $ax + by + cz = k$ with $k = \boldsymbol{n} \cdot \boldsymbol{a}$.

EXAMPLE · the triangle's plane

$A = (1, 0, 0)$, $\boldsymbol{n} = (2, 0, 1)$:
$2(x - 1) + (z - 0) = 0$, i.e. $\boxed{2x + z = 2}$.
Check B: $2 + 0 = 2$. Check C: $0 + 2 = 2$.



3D schematic

No y in the equation: y is *free*, the plane contains every line parallel to the y-axis through one of its points. It is not "$y = 0$".

Reading and writing plane equations

PROPOSITION · the coefficients are a normal

In $ax + by + cz = k$, $(a, b, c) \neq \mathbf{0}$ is a normal. Multiplying the whole equation by $\lambda \neq 0$ keeps the plane.

$4x + 2z = 4$ is the plane $2x + z = 2$;
$4x + 2z = 2$ is a parallel plane;
$0x + 0y + 0z = k$ is no plane.

PROPOSITION · two directions instead of a normal

If $\boldsymbol{u} \times \boldsymbol{v} \neq \mathbf{0}$, the plane through A with normal $\boldsymbol{u} \times \boldsymbol{v}$ is $\{A + s\boldsymbol{u} + r\boldsymbol{v} : \ s, r \in \mathbb{R}\}$.

Ours: $(1 - r, 2s, 2r)$; indeed $2(1 - r) + 2r = 2$.

Three non-collinear points give a plane: two edge vectors, their cross product as normal, one point.

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Lines and planes

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A line meets a plane: substitute

$\ell:\ X = P + t\boldsymbol{d},\ \Pi:\ \boldsymbol{n}\cdot\boldsymbol{x} = k\ \ (\boldsymbol{d}, \boldsymbol{n} \neq \boldsymbol{0})$. Put X into Π:

$$\underbrace{(\boldsymbol{n}\cdot\boldsymbol{d})}_{a}\, t = \underbrace{k - \boldsymbol{n}\cdot\boldsymbol{p}}_{b}.$$

ONE POINT

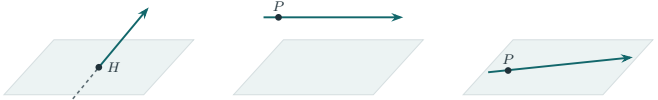
$a \neq 0$: $t = b/a$

NO POINT

$a = 0, b \neq 0$

EVERY POINT

$a = b = 0$: $\ell \subset \Pi$



one point — no point — the whole line

For a ray keep only $t \geq 0$; for a segment $P + t(Q - P)$ keep only $0 \leq t \leq 1$.

Does the ray reach the plane?

YOUR TURN · three minutes, in pairs

$\Pi:\ 2x + z = 2$, ray $(\frac{1}{2}, \frac{1}{2}, 3) + t(0, 0, -1)$, $t \geq 0$. Where does it hit? Would the segment $0 \leq t \leq 1$? The reversed ray?

EXAMPLE · answers

$1 + (3 - t) = 2$: $t = 2$, $H = (\frac{1}{2}, \frac{1}{2}, 1)$. Segment: no, $2 \notin [0, 1]$. Reversed: $t = -2$, no.

EXAMPLE · $\boldsymbol{n} \cdot \boldsymbol{d} = 0$

$\boldsymbol{d} = (1, 0, -2)$: $\boldsymbol{n} \cdot \boldsymbol{d} = 0$.
From $A = (1, 0, 0)$: $0 \cdot t = 0$, the line lies in Π.
From O: $0 \cdot t = 2$, no common point.

Rule: look at $\boldsymbol{n} \cdot \boldsymbol{d}$ before dividing by it. Zero is information, not a failure.

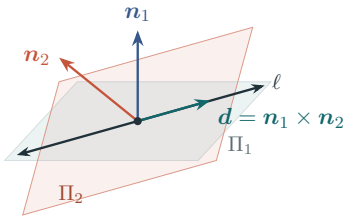
Two planes: look at the normals first

$\Pi_1 : \ \boldsymbol{n}_1 \cdot \boldsymbol{x} = k_1, \ \Pi_2 : \ \boldsymbol{n}_2 \cdot \boldsymbol{x} = k_2.$

- $\boldsymbol{n}_2 = \lambda \boldsymbol{n}_1$: the same plane if $k_2 = \lambda k_1$, otherwise parallel, disjoint.
- Otherwise a line, with direction $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$: it is perpendicular to *both* normals.

EXAMPLE

$2x + z = 2$, $x + y + z = 3$:
$\boldsymbol{d} = (2, 0, 1) \times (1, 1, 1) = (-1, -1, 2)$;
$z = 0$ gives the point $(1, 2, 0)$. So
$\Pi_1 \cap \Pi_2 : \ (1, 2, 0) + t(-1, -1, 2)$.



3D schematic

Check both equations for every t. A direction alone is not a line: you also need a point.

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Angles

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The angle between two planes

DEFINITION · angle between planes

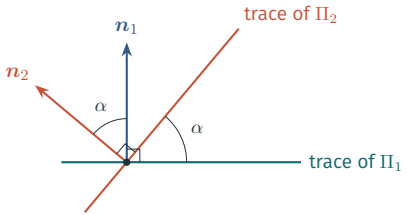
For normals $\boldsymbol{n}_1, \boldsymbol{n}_2 \neq \mathbf{0}$:

$$\cos\alpha = \frac{|\boldsymbol{n}_1 \cdot \boldsymbol{n}_2|}{\|\boldsymbol{n}_1\|\,\|\boldsymbol{n}_2\|}, \quad 0 \le \alpha \le \tfrac{\pi}{2}.$$

EXAMPLE

$2x + z = 2$ and the floor $z = 0$:

$$\cos\alpha = \frac{|(2,0,1)\cdot(0,0,1)|}{\sqrt{5}\cdot 1} = \frac{1}{\sqrt{5}}.$$



a section perpendicular to $\Pi_1 \cap \Pi_2$; equal scales

Why normals: in a section perpendicular to $\Pi_1 \cap \Pi_2$, each normal is its trace turned through a right angle. The absolute value: a normal has no preferred sign.

The angle between a line and a plane

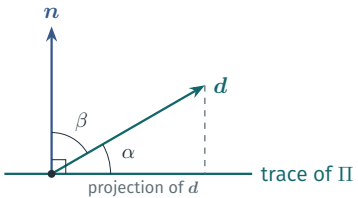
DEFINITION · angle between a line and a plane

Direction $\boldsymbol{d} \neq \mathbf{0}$, normal $\boldsymbol{n} \neq \mathbf{0}$:

$$\sin\alpha = \frac{|\boldsymbol{d}\cdot\boldsymbol{n}|}{\|\boldsymbol{d}\|\ \|\boldsymbol{n}\|}, \quad 0 \le \alpha \le \frac{\pi}{2}.$$

YOUR TURN · one minute

$\boldsymbol{d}\cdot\boldsymbol{n} = 0$: $\alpha =$? $\boldsymbol{d} \parallel \boldsymbol{n}$: $\alpha =$? A vertical line against $2x + z = 2$? **Answer:** 0; $\frac{\pi}{2}$; $\sin\alpha = \frac{1}{\sqrt{5}}$.



$\alpha + \beta = \pi/2$, so $\sin\alpha = \cos\beta$

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Area and volume

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Volume: base area times height

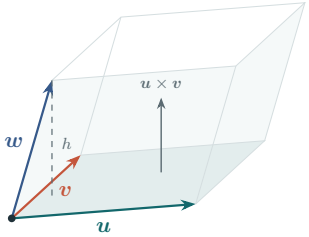
DEFINITION · scalar triple product

$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ — a number.

Base area $\|\boldsymbol{u} \times \boldsymbol{v}\|$, height $h = |\boldsymbol{w} \cdot \widehat{\boldsymbol{n}}|$, $\widehat{\boldsymbol{n}}$ the unit normal of the base:
$V_{\text{parallelepiped}} = \left|[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]\right|$.

TETRAHEDRON

$V = \frac{1}{6}\left|[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}]\right|$: half the base, and a pyramid is a third of its prism.



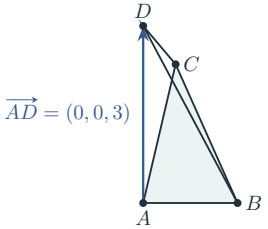
$V = (\text{base area}) \times h = \|\boldsymbol{u} \times \boldsymbol{v}\| \left|\boldsymbol{w} \cdot \widehat{\boldsymbol{n}}\right|$
3D schematic

The sign says on which side of the base $\boldsymbol{w}$ ends; a volume is the absolute value.

One more vertex: a volume and a test

Add $D = (1, 0, 3)$: $\overrightarrow{AD} = (0, 0, 3)$. $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = (4, 0, 2) \cdot (0, 0, 3) = 6$, so $V = \frac{1}{6} \cdot 6 = 1$. Check: the base has area $\sqrt{5}$, and D is $\overrightarrow{AD} \cdot \widehat{n} = \frac{3}{\sqrt{5}}$ above its plane (not 3: the base is tilted); so $V = \frac{1}{3}\sqrt{5} \cdot \frac{3}{\sqrt{5}} = 1$.

Four points are coplanar exactly when their triple product is 0; three points are collinear exactly when the cross product of two edges is $\mathbf{0}$.



3D schematic; base ABC, apex $D = (1, 0, 3)$

6

From the plane to the face

IN THIS LECTURE

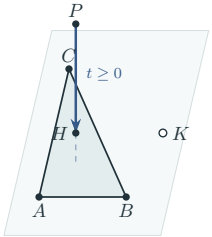
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Hitting the plane is only the first test

A mesh face is the *triangle* ABC, not its whole plane.

APPLICATION · ray against a triangle

1. Plane: normal $\overrightarrow{AB} \times \overrightarrow{AC}$, through A.
2. Hit: solve $at = b$; keep $t \geq 0$. Call the point H.
3. Face: $\overrightarrow{AH} = s\,\overrightarrow{AB} + r\,\overrightarrow{AC}$ with $s, r \geq 0$, $s + r \leq 1$.



3D schematic

Why: $H = (1 - s - r)A + sB + rC$ is a combination of the vertices with weights adding up to 1 (Lecture 1), and H is in the triangle exactly when all three weights are ≥ 0.

Finish our ray

The ray $(\frac{1}{2}, \frac{1}{2}, 3) + t(0, 0, -1)$ hit the plane at $H = (\frac{1}{2}, \frac{1}{2}, 1)$. Is H on the face?

$$\overrightarrow{AH} = (-\tfrac{1}{2}, \tfrac{1}{2}, 1) = s\,(0, 2, 0) + r\,(-1, 0, 2): \qquad r = \tfrac{1}{2}, \quad s = \tfrac{1}{4}, \quad 2r = 1 \checkmark$$

EXAMPLE · H

$s = \frac{1}{4}$, $r = \frac{1}{2}$, $s + r = \frac{3}{4} \leq 1$: on the face, strictly inside.

EXAMPLE · $K = (\frac{1}{2}, \frac{5}{2}, 1)$

In the plane ($1 + 1 = 2$), but $s = \frac{5}{4}$, $r = \frac{1}{2}$, $s + r > 1$: outside the face.

Plane hit, domain of the parameter, then the weights: three tests, in this order.

Exit check: diagnose the claim

YOUR TURN · two minutes, one sentence each

1. "I halved my normal, so the triangle's area is halved."
2. "The plane hit has $t = -3$, so the forward ray hits the plane."
3. "The triple product is -12, so the tetrahedron has volume -2."
4. "The normals $(1, 1, 0)$ and $(2, 2, 0)$ are parallel, so the two planes coincide."

All four are false: the area comes from the cross product of the edges; $t = -3$ is behind the start of the ray; a volume is an absolute value, here 2; parallel normals give parallel planes, which coincide only if the constants match.

Summary

- $\boldsymbol{u} \times \boldsymbol{v}$: perpendicular to both, length $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\| \sin\theta$ (a parallelogram's area), right-handed; $\mathbf{0}$ exactly for collinear vectors.
- Plane through A with normal $\boldsymbol{n}$: $\boldsymbol{n} \cdot \overrightarrow{AX} = 0$, i.e. $ax + by + cz = k$; through three points: normal $\overrightarrow{AB} \times \overrightarrow{AC}$.
- Line against plane: substitute, read $(\boldsymbol{n} \cdot \boldsymbol{d})\, t = k - \boldsymbol{n} \cdot \boldsymbol{p}$, then the domain of t. Two planes: parallel normals, or a line with direction $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ and one point.
- Angles are the smaller ones: cosine between two normals, sine for a line against a plane.
- $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$: signed volume; a tetrahedron is one sixth; zero means coplanar.
- A ray hits a face when it hits the plane with $t \geq 0$ *and* the weights $s, r, 1 - s - r$ are all ≥ 0.

WRAP-UP

Seminar 2, second meeting: what to prepare

Prepare: every part of Problems 2.2 and 2.3 (Core). You may be asked to present them at the board.

- Also for this meeting: Problems 2.5 and 2.6 (Medium), the board problems 2.B6–2.B8 and the Challenge 2.7 — choices for discussion, not homework to finish.

NEXT LECTURE

Distances and reflections: how far is a point from a plane, and on which side is it? A normal measures it.