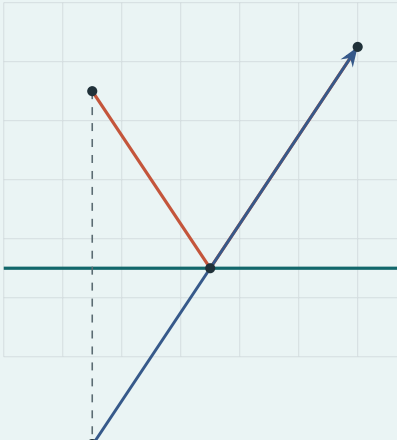
ANALYTIC GEOMETRY · LECTURE 5

Distances, reflections and shortest paths

How far is a point from a line, where is its mirror image, and where should a shortest route touch the mirror?



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Babeș–Bolyai University · Mathematics and Computer Science · 2026–2027

Three questions for today

YOUR TURN · how far?

A point and a line (or a plane): how far apart are they, and which point of the line is the closest?

YOUR TURN · the mirror image

A line is a mirror. Where does it send a point? And where does it send a *direction*?

YOUR TURN · the shortest detour

From A to a straight river bank, then on to B. Where should you touch the bank?

One construction answers all three: the *perpendicular* from the point to the line. Coordinates today: an orthonormal Cartesian frame with a common unit, in the plane or in space.

Quick check

YOUR TURN · two minutes, then compare with your neighbour

1. $\ell:\ x + 2y = 4$. A normal vector and a direction vector of ℓ? Is $H = (2, 1)$ on ℓ? Is $P = (4, 5)$?
2. $\Pi:\ 2x - y + 2z = 1$. A normal vector? Is $(1, 1, 0)$ in Π? Is $(3, 0, 2)$?

EXAMPLE · answers

$\boldsymbol{n} = (1, 2)$, the coefficients; $\boldsymbol{d} = (2, -1)$, since $\boldsymbol{n} \cdot \boldsymbol{d} = 0$. H: $2 + 2 = 4$, yes. P: $4 + 10 \neq 4$, no.
$\boldsymbol{n} = (2, -1, 2)$. $(1, 1, 0)$: $2 - 1 + 0 = 1$, yes. $(3, 0, 2)$: $6 + 4 = 10 \neq 1$, no.

A line in the plane and a plane in space have the same kind of equation, $\boldsymbol{n} \cdot \boldsymbol{x} = c$ with $\boldsymbol{n} \neq \boldsymbol{0}$. Everything today works for both at once.

1

The foot of the perpendicular

IN THIS LECTURE

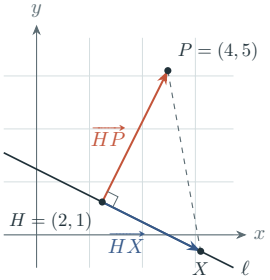
1 The foot of the perpendicular
2 Signed distance: which side?
3 Reflections
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5 An application: how far from the boundary?

Follow the normal from P to the line

$E:\ \boldsymbol{n}\cdot\boldsymbol{x}=c$ with $\boldsymbol{n}\neq\mathbf{0}$; a point P with position vector $\boldsymbol{p}$.

The foot H is on the normal line through P: $H=P-\alpha\boldsymbol{n}$.

$$\boldsymbol{n}\cdot(\boldsymbol{p}-\alpha\boldsymbol{n})=c \quad\Longrightarrow\quad \alpha=\frac{\boldsymbol{n}\cdot\boldsymbol{p}-c}{\|\boldsymbol{n}\|^2}.$$



THE FOOT

$$H=P-\frac{\boldsymbol{n}\cdot\boldsymbol{p}-c}{\|\boldsymbol{n}\|^2}\,\boldsymbol{n}$$

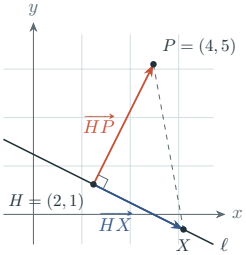
$\boldsymbol{n}\cdot\boldsymbol{p}-c$ is the *residual*: how badly P fails the equation. It is 0 exactly when $P\in E$, and then $H=P$.

One example, three checks

$\ell:\ x + 2y = 4$, $P = (4, 5)$. $\boldsymbol{n} = (1, 2)$, $\|\boldsymbol{n}\|^2 = 5$, residual $4 + 10 - 4 = 10$, so $\alpha = 2$ and $H = (4, 5) - 2\,(1, 2) = (2, 1)$.

YOUR TURN · one minute: verify all three

1. H on ℓ? $2 + 2 \cdot 1 = 4$. ✓
2. $PH \perp \ell$? $\overrightarrow{HP} = (2, 4) = 2\boldsymbol{n}$. ✓
3. $PH = ?$ $\|(2, 4)\| = \sqrt{20} = 2\sqrt{5}$.



Careful: the residual 10 is *not* the distance. Multiply the equation by 10: the residual becomes 100, yet the line has not moved.

Why no other point of the line is closer

THEOREM · closest point

For every point X of E: $PX^2 = PH^2 + HX^2$. So H is the *unique* closest point, and $d(P, E) = PH = \dfrac{|\boldsymbol{n} \cdot \boldsymbol{p} - c|}{\|\boldsymbol{n}\|}$.

1. $\overrightarrow{PX} = \overrightarrow{PH} + \overrightarrow{HX}$, where $\overrightarrow{PH} = \alpha \boldsymbol{n}$ is along the normal.
2. H and X both lie in E, so $\overrightarrow{HX} \perp \boldsymbol{n}$, i.e. $\overrightarrow{PH} \cdot \overrightarrow{HX} = 0$.
3. $PX^2 = \left\|\overrightarrow{PH}\right\|^2 + 2\,\overrightarrow{PH} \cdot \overrightarrow{HX} + \left\|\overrightarrow{HX}\right\|^2 = PH^2 + HX^2 \geq PH^2$; equality only for $X = H$.
4. $PH = |\alpha|\ \|\boldsymbol{n}\| = \dfrac{|\boldsymbol{n} \cdot \boldsymbol{p} - c|}{\|\boldsymbol{n}\|^2}\ \|\boldsymbol{n}\|$. $\square$

As in Lecture 3: Pythagoras for *every* X at once.

The same calculation in space

YOUR TURN · two minutes

$\Pi:\ 2x - y + 2z = 1$ and $P = (3, 0, 2)$. Find the foot H and the distance $d(P, \Pi)$; check H in the equation.

EXAMPLE · answer

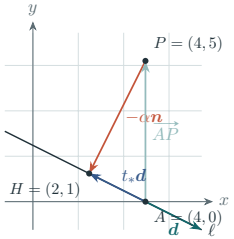
$\boldsymbol{n} = (2, -1, 2)$, $\|\boldsymbol{n}\|^2 = 9$, residual $6 - 0 + 4 - 1 = 9$, so $\alpha = 1$:
$H = (3, 0, 2) - (2, -1, 2) = (1, 1, 0)$ and $d(P, \Pi) = \frac{9}{3} = 3$.
Check: $2 - 1 + 0 = 1$. ✓ And $\overrightarrow{HP} = (2, -1, 2) = \boldsymbol{n}$, of length 3.

Same formula, same proof: nothing depended on the number of coordinates.

A second route: project onto the direction

If the line is given as $A + t\boldsymbol{d}$ ($\boldsymbol{d} \neq \mathbf{0}$), Lecture 3 found its closest point to P:

$$t_* = \frac{\overrightarrow{AP} \cdot \boldsymbol{d}}{\boldsymbol{d} \cdot \boldsymbol{d}}, \qquad H = A + t_* \boldsymbol{d}.$$



EXAMPLE · same line, same foot

ℓ through $A = (4, 0)$ with $\boldsymbol{d} = (2, -1)$;
$\overrightarrow{AP} = (0, 5)$.
$t_* = \frac{0-5}{5} = -1$,
$H = (4, 0) - (2, -1) = (2, 1)$. ✓

A line in *space* has no normal equation: there, project. On a ray or segment, take the allowed t nearest to t_*.

2

Signed distance: which side?

IN THIS LECTURE

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2 Signed distance: which side?
3 Reflections
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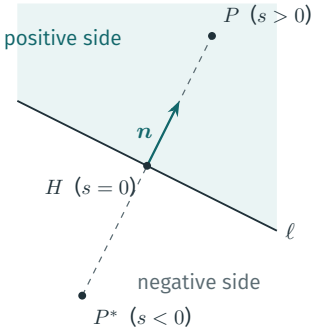
A sign records the side

Let $\widehat{\boldsymbol{n}} = \boldsymbol{n}/\,\|\boldsymbol{n}\|$ be the unit normal.

DEFINITION · signed distance

$$s(P) = \frac{\boldsymbol{n}\cdot\boldsymbol{p} - c}{\|\boldsymbol{n}\|}; \quad \text{then } \overrightarrow{HP} = s(P)\,\widehat{\boldsymbol{n}}$$

and $d(P, E) = |s(P)|$.



$s > 0$: P is on the side $\boldsymbol{n}$ points to. $s < 0$: the other side. $s = 0$: $P \in E$.

For $x + 2y = 4$: $s(4, 5) = \frac{10}{\sqrt{5}} = 2\sqrt{5}$, $s(0, -3) = -2\sqrt{5}$, $s(2, 1) = 0$.

To travel from P *to* the line, use $\overrightarrow{PH} = -s(P)\,\widehat{\boldsymbol{n}}$: the shortest displacement onto E.

Predict before you calculate

YOUR TURN · one minute

At $P = (4, 5)$, compare $x + 2y = 4$, $3x + 6y = 12$ and $-x - 2y = -4$. Same foot, distance, signed distance?

One line, three names: the foot $(2, 1)$ and the distance $2\sqrt{5}$ cannot change. Signed distances: $2\sqrt{5}$, $\frac{30}{\sqrt{45}} = 2\sqrt{5}$, $\frac{-10}{\sqrt{5}} = -2\sqrt{5}$.

PROPOSITION · rescaling the whole equation

Replacing $(\boldsymbol{n}, c)$ by $(\lambda\boldsymbol{n}, \lambda c)$, $\lambda \neq 0$, keeps the set, the foot and the distance; the signed distance is multiplied by $\lambda/|\lambda| = \pm 1$.

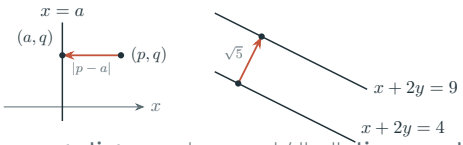
Two cases a slope formula would hide

EXAMPLE · a vertical line

$x = a$ has normal $(1, 0)$. From $P = (p, q)$: $H = (a, q)$, $d = |p - a|$. No slope, no special case.

EXAMPLE · two parallel lines

$x + 2y = 4$ and $2x + 4y = 18$. Give them the *same* normal first: $x + 2y = 9$. Then $d = \frac{|9-4|}{\sqrt{5}} = \sqrt{5}$.



$\boldsymbol{n} \cdot \boldsymbol{x} = c_1$ and $\boldsymbol{n} \cdot \boldsymbol{x} = c_2$ are at distance $|c_1 - c_2| / \|\boldsymbol{n}\|$, lines and planes alike. $0x + 0y = c$ is not a line.

3

Reflections

IN THIS LECTURE

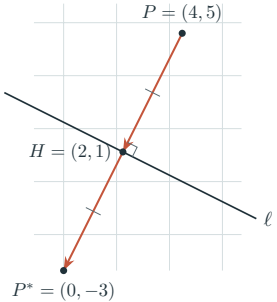
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The mirror image of a point

The foot H is the midpoint of P and its image P^*: $P^* = 2H - P$.

REFLECTION OF A POINT

$$P^* = P - 2\,\frac{\boldsymbol{n}\cdot\boldsymbol{p} - c}{\|\boldsymbol{n}\|^2}\,\boldsymbol{n}$$



$P = (4, 5)$ in $x + 2y = 4$:
$P^* = (4, 5) - 4\,(1, 2) = (0, -3)$.
Check: midpoint $(2, 1) \in \ell$, and $\overrightarrow{PP^*} = (-4, -8) \parallel \boldsymbol{n}$.

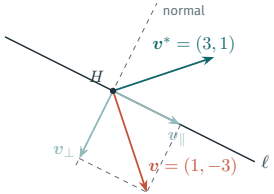
Twice the displacement that reaches the line. Points of the mirror stay put; reflecting twice gives P back. The frame does not move, the point does.

The mirror image of a direction

Split $\boldsymbol{v}$ into $\boldsymbol{v}_\perp = \dfrac{\boldsymbol{v}\cdot\boldsymbol{n}}{\|\boldsymbol{n}\|^2}\,\boldsymbol{n}$ along the normal and $\boldsymbol{v}_\parallel = \boldsymbol{v} - \boldsymbol{v}_\perp$ along the mirror.

REFLECTION OF A DIRECTION

$$\boldsymbol{v}^* = \boldsymbol{v}_\parallel - \boldsymbol{v}_\perp = \boldsymbol{v} - 2\,\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{\|\boldsymbol{n}\|^2}\,\boldsymbol{n}$$



$\boldsymbol{n} = (1,2)$, $\boldsymbol{v} = (1,-3)$: $\boldsymbol{v}\cdot\boldsymbol{n} = -5$,
$\boldsymbol{v}^* = (1,-3) + 2\,(1,2) = (3,1)$.

A point has a place, a direction has none: that is why c appears in one formula and not in the other.

Your turn: the mirror $x = 1$

YOUR TURN · two minutes

Reflect the point $P = (3, 4)$ and the direction $\boldsymbol{v} = (1, 2)$ in the line $x = 1$. Then write the image of the ray $P + t\boldsymbol{v}$, $t \geq 0$.

EXAMPLE · answers

$\boldsymbol{n} = (1, 0)$, $c = 1$. Point: residual 2, so $P^* = (3, 4) - 4\,(1, 0) = (-1, 4)$.
Direction: $\boldsymbol{v} \cdot \boldsymbol{n} = 1$, so $\boldsymbol{v}^* = (1, 2) - 2\,(1, 0) = (-1, 2)$: the x-component flips, the y-component stays.
Ray: $P^* + t\boldsymbol{v}^* = (-1, 4) + t(-1, 2)$, $t \geq 0$, the same t.

Careful: the *point* $(1, 2)$ lies on the mirror and stays there. That is not the reflected direction.

Reflections preserve distances

PROPOSITION

$\|\boldsymbol{v}^*\| = \|\boldsymbol{v}\|$ for every direction, and $P^*Q^* = PQ$ for every two points: a reflection is an *isometry*.

1. $\boldsymbol{v}_{\parallel} \perp \boldsymbol{v}_{\perp}$, so by Pythagoras $\left\|\boldsymbol{v}_{\parallel} - \boldsymbol{v}_{\perp}\right\|^2 = \left\|\boldsymbol{v}_{\parallel}\right\|^2 + \|\boldsymbol{v}_{\perp}\|^2 = \left\|\boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}\right\|^2$.
2. Subtract the point formulas for Q and P: the constants c cancel, and what remains is the direction formula applied to $\overrightarrow{PQ}$. So $\overrightarrow{P^*Q^*} = (\overrightarrow{PQ})^*$, of the same length. $\square$

A reflection moves a figure without distorting it. Lecture 6 makes “without distorting” a definition and gives a test for it: look at the columns of a matrix.

4

The shortest detour

IN THIS LECTURE

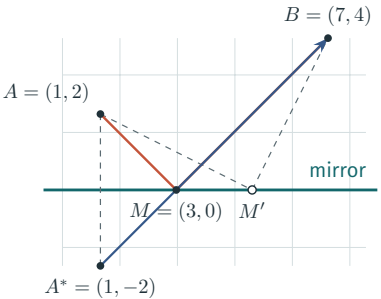
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Where should the route touch the mirror?

A, B on the same side of ℓ; for $M \in \ell$ the route $A \to M \to B$ has length $AM + MB$. Which M makes it shortest?

Reflect A to A^*. The reflection fixes M and keeps distances, so $AM = A^*M$ for *every* M on ℓ.

So $AM + MB = A^*M + MB \geq A^*B$, with equality exactly when $M \in [A^*B]$.



Heron of Alexandria asked this about light and a mirror almost two thousand years ago. The answer has not changed: unfold the route into a straight line.

From a lower bound to a minimum

THEOREM · shortest route touching a line

If A, B lie strictly on the same side of ℓ, the shortest route $A \to M \to B$ with $M \in \ell$ is unique: M is where $[A^*B]$ crosses ℓ, and the minimum length is A^*B.

1. **Bound.** $AM + MB \geq A^*B$ for every $M \in \ell$.
2. **Equality.** Only when $M \in [A^*B]$: the equality case of the triangle inequality (Lecture 3).
3. **Attained.** A^* and B are on opposite sides, so $[A^*B]$ crosses ℓ exactly once; that point is allowed and attains A^*B. $\square$

Careful: a lower bound is half a proof. "At least A^*B" becomes "the minimum is A^*B" only once an allowed M attains it.

Find the contact point

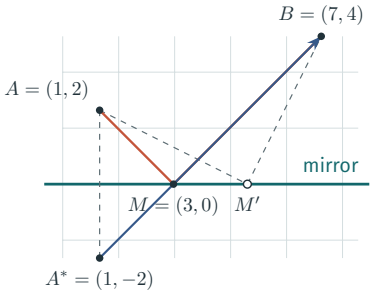
EXAMPLE · $A = (1, 2)$, $B = (7, 4)$, mirror $y = 0$

$A^* = (1, -2)$. Segment: $(1, -2) + t\,(6, 6)$, $0 \le t \le 1$.

$y = 0$ at $t = \frac{1}{3} \in [0, 1]$: $M = (3, 0)$ is allowed.

Minimum: $A^*B = \|(6, 6)\| = 6\sqrt{2}$.

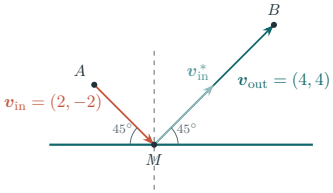
Fold back: $AM = \|(2, -2)\| = 2\sqrt{2}$, $MB = \|(4, 4)\| = 4\sqrt{2}$, sum $6\sqrt{2}$. The dashed route through $M' = (5, 0)$ is longer: $\sqrt{20} + \sqrt{20} \approx 8.94 > 8.49$.



Equal angles: the law of reflection falls out

On the shortest route A^*, M, B are collinear, with M between them. The reflection sends $A^* \mapsto A$ and fixes M, so $\overrightarrow{AM}$ is the mirror image of $\overrightarrow{A^*M}$, which points along $\overrightarrow{MB}$.

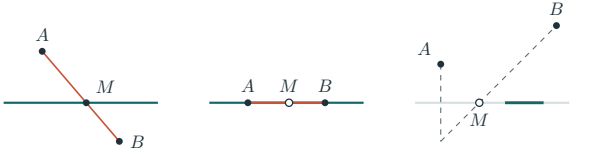
The outgoing direction $\overrightarrow{MB}$ is the reflection of the incoming $\overrightarrow{AM}$, up to a positive factor: angle in equals angle out.



$\overrightarrow{AM} = (2, -2)$ reflected in $y = 0$ is $(2, 2)$, and $\overrightarrow{MB} = (4, 4) = 2\,(2, 2)$. ✓

Light at a flat mirror takes the shortest path. A bounce starts at the impact point: the reflected ray is $M + t\,(\overrightarrow{AM})^*$ with $t \geq 0$.

When the picture changes, the argument changes



opposite sides: $[AB]$ already crosses ℓ

both on the mirror: every M between them

part of the mirror: the equality point is forbidden

YOUR TURN · what is the shortest route now?

1. A and B on opposite sides of ℓ. 2. Both A and B on ℓ. 3. Our example, but only $4 \le x \le 6$ of the mirror may be used.

1. No reflection: $AM + MB \ge AB$, attained where $[AB]$ crosses ℓ. 2. Every M between A and B gives AB. 3. $M = (3, 0)$ is forbidden: $6\sqrt{2}$ is a bound that is *not attained*, a new problem (3.4).

5

An application: how far from the boundary?

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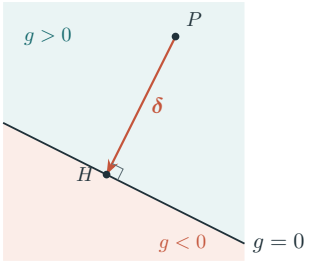
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A score is not a distance

A rule classifies points by the sign of $g(x, y) = ax + by - c$, with $(a, b) \neq (0, 0)$. The boundary is the line $g = 0$; the value $g(P)$ is a *score*.

APPLICATION · the smallest change that reaches the boundary

$$d = \frac{|g(P)|}{\sqrt{a^2 + b^2}}, \qquad \boldsymbol{\delta} = -\frac{g(P)}{a^2 + b^2}\,(a, b).$$



$g = x + 2y - 4$ at $P = (4, 5)$: score 10, distance $2\sqrt{5}$, $\boldsymbol{\delta} = (-2, -4)$.

Replacing g by $10g$ multiplies every score by 10 and moves nothing: d and $\boldsymbol{\delta}$ are unchanged. A distance means something only if both coordinates use the same unit.

Exit check: diagnose the claim

YOUR TURN · two minutes, one sentence each

1. "$x + 2y - 4$ equals 10 at P, so P is 10 units from the line."
2. "Multiplying the equation of the mirror by -3 moves the foot of the perpendicular."
3. "To reflect the direction $(1, 2)$ in $x = 1$, I reflect the point $(1, 2)$ and get $(1, 2)$."
4. "$AM + MB \geq A^*B$ for all M, so the minimum is A^*B."

All four are false: divide the residual by $\|\boldsymbol{n}\| = \sqrt{5}$; rescaling changes only the sign of the signed distance; a direction needs the direction formula, $(-1, 2)$; a bound is a minimum only when an allowed M attains it.

Summary

- Foot of the perpendicular from P to $\boldsymbol{n} \cdot \boldsymbol{x} = c$: $H = P - \frac{\boldsymbol{n}\cdot\boldsymbol{p} - c}{\|\boldsymbol{n}\|^2}\boldsymbol{n}$; for a line $A + t\boldsymbol{d}$, project $\overrightarrow{AP}$ onto $\boldsymbol{d}$.
- $PX^2 = PH^2 + HX^2$ for every X of E: H is the unique closest point, and $d(P, E) = \frac{|\boldsymbol{n}\cdot\boldsymbol{p} - c|}{\|\boldsymbol{n}\|}$.
- Signed distance $s = \frac{\boldsymbol{n}\cdot\boldsymbol{p} - c}{\|\boldsymbol{n}\|}$: its sign is the side chosen by $\boldsymbol{n}$; rescaling the equation changes at most that sign.
- Reflection: $P^* = P - 2s\,\widehat{\boldsymbol{n}}$ for a point, $\boldsymbol{v}^* = \boldsymbol{v} - 2\frac{\boldsymbol{v}\cdot\boldsymbol{n}}{\|\boldsymbol{n}\|^2}\boldsymbol{n}$ for a direction (no c); distances are preserved.
- Shortest route touching a line: reflect one endpoint, join, fold back; bound, equality case, attained.

WRAP-UP

Seminar 3: one sheet for two lectures

Prepare: every part of Problem 3.1 (Core). You may be asked to present it at the board.

- Also for this meeting: Problem 3.4 (Medium) and the board problems 3.B1–3.B5 — choices for discussion, not homework to finish.

AFTER LECTURE 6

Problems 3.2 and 3.3 (Core), 3.5 and 3.6 (Medium), 3.B6–3.B8; the Challenge 3.7 uses both lectures.

NEXT LECTURE

Isometries and affine maps: matrices as the language for rotations, reflections and distortions.