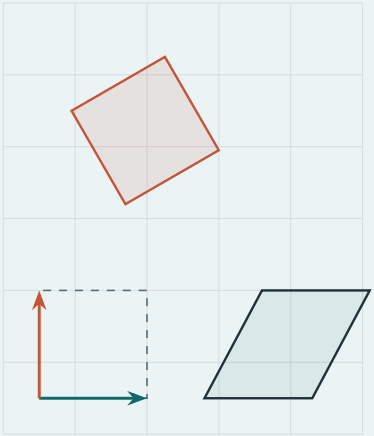
ANALYTIC GEOMETRY · LECTURE 6

Isometries and affine transformations

Which maps move a figure without distorting it, and which only keep its lines straight?



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Three questions for today

YOUR TURN · moving a figure

A square is rotated, shifted, or stretched sideways. Which rule gives the new position of each of its points?

YOUR TURN · undoing it

The positions were changed by a known rule. Can we recover the originals, and when is that impossible?

YOUR TURN · what survives?

Which rules keep lengths and angles, like the reflections of Lecture 5, and which only keep lines straight?

New language today: matrices, introduced as far as we need them and no further. A tuple (x, y) becomes a *column* when a matrix acts on it. Fixed orthonormal axes with a common unit, as always.

Quick check

YOUR TURN · two minutes, then compare with your neighbour

1. Translate $P = (1, 0)$ and $Q = (1, 2)$ by $\boldsymbol{b} = (2, 1)$. Where are P' and Q'? What happens to $\overrightarrow{PQ}$, and to its length?
2. Reflect the point $(3, 4)$ and the direction $(1, 2)$ in the line $x = 1$.

EXAMPLE · answers

$P' = (3, 1)$, $Q' = (3, 3)$; $\overrightarrow{P'Q'} = (0, 2) = \overrightarrow{PQ}$, still of length 2: a translation moves points, not displacements.
$(3, 4) \mapsto (-1, 4)$ with the point formula; $(1, 2) \mapsto (-1, 2)$ with the direction formula.

1

A matrix is two arrows

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Where do the unit arrows go?

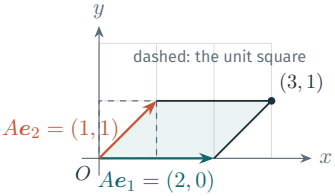
A rule sends $\boldsymbol{e}_1 = (1,0)$ to $(2,0)$ and $\boldsymbol{e}_2 = (0,1)$ to $(1,1)$ and respects sums and multiples. Then it sends

$$x\boldsymbol{e}_1 + y\boldsymbol{e}_2 \longmapsto x\,(2,0) + y\,(1,1) = (2x+y,\ y).$$

DEFINITION · store the two images as columns

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix},$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x+y \\ y \end{pmatrix}.$$



Multiply by combining the columns

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

YOUR TURN · one minute

For $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, compute $A(1,1)$ and $A(2,-1)$ by combining the columns.

Answers: $(2,0) + (1,1) = (3,1)$ and $2\,(2,0) - (1,1) = (3,-1)$.

The action is linear: $A(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha A\boldsymbol{u} + \beta A\boldsymbol{v}$; in particular $A\mathbf{0} = \mathbf{0}$.

A translation by $\boldsymbol{b} \neq \mathbf{0}$ sends $\mathbf{0}$ to $\boldsymbol{b}$, so it is not of this form: one more ingredient is needed.

Place the shape: add a translation

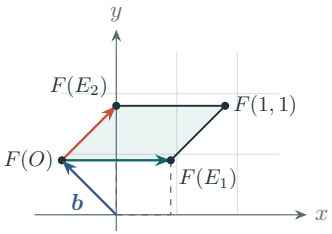
DEFINITION · affine map

$F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. The matrix A shapes the figure; $\boldsymbol{b} = F(O)$ places it.

With $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{b} = (-1, 1)$:

$$F(x, y) = (2x + y - 1,\ y + 1).$$

An *active* map: the points move, the axes stay. This includes the translations ($A = I$) and the reflections of Lecture 5.



Calibrate from three reference points

APPLICATION · a grid photographed at an angle

Three marked points were measured: $O \mapsto (-1, 1)$, $E_1 = (1, 0) \mapsto (1, 1)$, $E_2 = (0, 1) \mapsto (0, 2)$. Which affine map did the camera apply?

1. The translation is the image of the origin: $\boldsymbol{b} = F(O) = (-1, 1)$.
2. The first column is the first edge: $A\boldsymbol{e}_1 = F(E_1) - F(O) = (2, 0)$.
3. The second column is the second edge: $A\boldsymbol{e}_2 = F(E_2) - F(O) = (1, 1)$.

YOUR TURN · one minute

So $F(x, y) = (2x + y - 1, y + 1)$. Where does the point $(2, 1)$ go? **Answer:** $(4, 2)$.

Points get the translation, displacements do not

Subtract two image points:

$$\overrightarrow{F(P)F(Q)} = (A\boldsymbol{q} + \boldsymbol{b}) - (A\boldsymbol{p} + \boldsymbol{b}) = A\,(\boldsymbol{q} - \boldsymbol{p}) = A\,\overrightarrow{PQ}.$$

EXAMPLE · the point $P = (2, 1)$

$$F(P) = (5, 1) + (-1, 1) = (4, 2).$$

EXAMPLE · the displacement $\overrightarrow{OP} = (2, 1)$

$A\,\overrightarrow{OP} = (5, 1)$; indeed $F(P) - F(O) = (5, 1)$.

YOUR TURN · a line $P + t\boldsymbol{d}$, with $\boldsymbol{d} \neq \boldsymbol{0}$ and $t \in \mathbb{R}$

Where does it go? **Answer:** to $F(P) + t\,A\boldsymbol{d}$; the point gets $\boldsymbol{b}$, the direction does not.

2

Combine, then undo

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Composition is substitution

$F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ and $G(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{d}$. First F, then G:

$$\boldsymbol{x} \xrightarrow{F} A\boldsymbol{x} + \boldsymbol{b} \xrightarrow{G} C(A\boldsymbol{x} + \boldsymbol{b}) + \boldsymbol{d}.$$

FIRST F, THEN G

$(G \circ F)(\boldsymbol{x}) = (CA)\,\boldsymbol{x} + (C\boldsymbol{b} + \boldsymbol{d})$: linear part CA, translation $C\boldsymbol{b} + \boldsymbol{d}$.

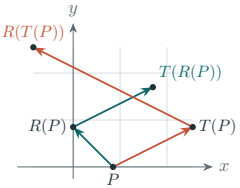
CA is the matrix with columns $C(A\boldsymbol{e}_1)$ and $C(A\boldsymbol{e}_2)$; writing them out gives the rows-times-columns rule. The map on the *right* acts first.
Careful: C also acts on the first translation $\boldsymbol{b}$. A translation done before a rotation gets rotated.

Rotate then translate, or translate then rotate?

$R(x, y) = (-y, x)$, the quarter-turn about O;
$T(x, y) = (x + 2, y + 1)$.

YOUR TURN · two minutes

For $P = (1, 0)$ compute $T(R(P))$ and $R(T(P))$; then find both maps in general.



EXAMPLE · answers

$$T(R(P)) = T(0, 1) = (2, 2),$$
$$R(T(P)) = R(3, 1) = (-1, 3).$$

In general $(T \circ R)(x, y) = (-y + 2,\ x + 1)$ and $(R \circ T)(x, y) = (-y - 1,\ x + 2)$: the same linear part R, translations $(2, 1)$ and $R(2, 1) = (-1, 2)$.

Undo in the reverse order

$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$ gives $\boldsymbol{x} = A^{-1}(\boldsymbol{x}' - \boldsymbol{b})$: subtract $\boldsymbol{b}$ first, then undo A.

THE PLANAR INVERSE

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\Delta = ad - bc \neq 0$:

$$A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

EXAMPLE · undo the calibration

$x' = 2x + y - 1$, $y' = y + 1$: solve from the bottom,

$$y = y' - 1, \qquad x = \tfrac{1}{2}(x' - y' + 2).$$

$(4, 2)$ goes back to $(2, 1)$. ✓

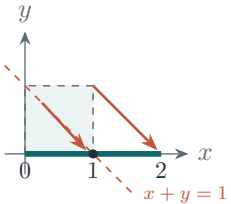
$A^{-1}A = AA^{-1} = I$, the identity, which does nothing. And $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$: undo the last step first.

When $\Delta = 0$, the map squashes the plane

$$C(x, y) = (x + y,\, 0),\ C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ \Delta = 0.$$

YOUR TURN · which points are sent to $(1, 0)$?

Every point of the line $x + y = 1$.



Different inputs, the same output: nothing can undo C. The whole plane lands on the x-axis.

A line $P + t\boldsymbol{d}$ collapses to the single point $F(P)$ exactly when $A\boldsymbol{d} = \mathbf{0}$; here $\boldsymbol{d} = (1, -1)$. With $\Delta \neq 0$ this never happens.

3

What survives?

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Affine maps keep lines straight and ratios fixed

PROPOSITION

For every affine map F, points P, Q and number t:
$F((1-t)P + tQ) = (1-t)F(P) + tF(Q)$.

- **Why:** $A((1-t)\boldsymbol{p} + t\boldsymbol{q}) + \boldsymbol{b} = (1-t)(A\boldsymbol{p} + \boldsymbol{b}) + t(A\boldsymbol{q} + \boldsymbol{b})$, since $(1-t) + t = 1$.
- $t = \frac{1}{2}$: midpoints to midpoints; weights adding up to 1: centroids to centroids.
- $0 \le t \le 1$: a segment to a segment (or a point); $t \in \mathbb{R}$: a line to a line (or a point).

If $\Delta \neq 0$: lines to lines, parallel lines to parallel lines, and division ratios along a line are kept. Lengths and angles need not be.

Areas: the determinant

DEFINITION · determinant

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$: the signed area of the parallelogram spanned by the two columns.

EXAMPLE · the shear $S(x, y) = (x + y,\ y)$

$\det S = 1$: areas are preserved.

But $(0, 1) \mapsto (1, 1)$: a length 1 becomes $\sqrt{2}$, and a right angle is lost.

Every area is multiplied by $|\det A|$; the sign says whether the orientation is kept ($+$) or reversed ($-$). Our calibration doubles areas.

Careful: $\det A = \pm 1$ keeps areas; it does not make the map rigid. Lengths need a different test.

Which maps keep distances? Test the columns

An **isometry** is a map that preserves the distance between every two points.

THEOREM · the distance test

$F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ is an isometry exactly when the columns of A are *orthonormal*: each of length 1, and perpendicular to each other.

YOUR TURN · one minute: which of these pass?

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Answer: the first, third and fourth. The shear fails: $(1, 1)$ has length $\sqrt{2}$.

Why orthonormal columns work

Let the columns $\boldsymbol{q}_1, \boldsymbol{q}_2$ have length 1 and $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$. For $\boldsymbol{v} = (x, y)$:

$$\|A\boldsymbol{v}\|^2 = \|x\boldsymbol{q}_1 + y\boldsymbol{q}_2\|^2 = x^2 \|\boldsymbol{q}_1\|^2 + 2xy\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + y^2 \|\boldsymbol{q}_2\|^2 = x^2 + y^2 = \|\boldsymbol{v}\|^2 .$$

Displacements transform by A alone, so every distance $PQ = \left\|\overrightarrow{PQ}\right\|$ is preserved, whatever $\boldsymbol{b}$ is.

The *transpose* A^{T} swaps rows and columns; the entries of $A^{\mathsf{T}}A$ are the dot products of the columns. So orthonormal columns means $A^{\mathsf{T}}A = I$, i.e. $A^{-1} = A^{\mathsf{T}}$: undoing an isometry costs nothing. Conversely, an isometry keeps $OE_1 = OE_2 = 1$ and $E_1E_2 = \sqrt{2}$, which forces unit columns with dot product 0 (notes).

Rotations, and a rotation about any centre

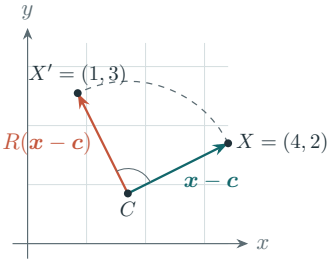
EXAMPLE · rotation by θ about O, anticlockwise

Columns: the unit arrows turned by θ,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

ROTATION ABOUT C

$\boldsymbol{x}' = \boldsymbol{c} + R_\theta(\boldsymbol{x} - \boldsymbol{c})$: move C to O, rotate, move it back.



$C = (2, 1)$, quarter-turn: $X = (4, 2)$ has $\boldsymbol{x} - \boldsymbol{c} = (2, 1)$, $R(2, 1) = (-1, 2)$, $X' = (1, 3)$. And C itself stays fixed. The reflection in the x-axis, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, also has orthonormal columns, with determinant -1.

4

Images and coordinate frames

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The equation of an image: substitute the inverse

YOUR TURN · a circle is translated by $(3, -2)$

Old circle $x^2 + y^2 = 4$; map $(x', y') = (x + 3,\ y - 2)$. Which equation describes the image?

$$x = x' - 3, \quad y = y' + 2 \quad \Longrightarrow \quad (x' - 3)^2 + (y' + 2)^2 = 4.$$

An image point (x', y') belongs to the image exactly when its *preimage* belongs to the old locus: write the inverse, substitute into the *old* equation.

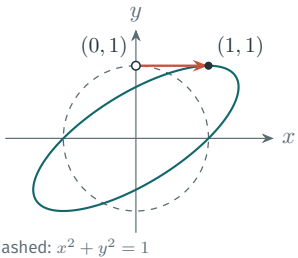
Substituting the forward map would describe the preimage of the circle, not its image.

Shear a circle, and let a point check the answer

$S(x, y) = (x + y,\ y)$; old circle $x^2 + y^2 = 1$.

YOUR TURN · two minutes

Find the equation of the image. Check it on the image of $(0, 1)$.



EXAMPLE · answer

Inverse: $x = x' - y'$, $y = y'$. Image: $(x' - y')^2 + y'^2 = 1$.
$(0, 1) \mapsto (1, 1)$, and $(1 - 1)^2 + 1 = 1$. ✓

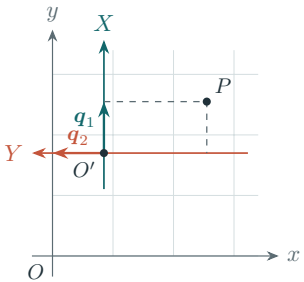
The forward substitution $(x' + y')^2 + y'^2 = 1$ fails at $(1, 1)$. A singular map has no inverse: then transform a parametrisation instead (notes).

Same point, new coordinates

Now nothing moves. New origin O' with old coordinates $\boldsymbol{o}$; new axes along orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$, the columns of Q.

DEFINITION · passive change of frame

$\boldsymbol{x} = \boldsymbol{o} + Q\boldsymbol{X}$, hence $\boldsymbol{X} = Q^{\mathsf{T}}(\boldsymbol{x} - \boldsymbol{o})$.
Old coordinates $\boldsymbol{x}$, new coordinates $\boldsymbol{X} = (X, Y)$, the same point.



Read it from the picture: to reach P from O', go X steps along $\boldsymbol{q}_1$ and Y steps along $\boldsymbol{q}_2$. This is *not* the active map $\boldsymbol{x} \mapsto Q\boldsymbol{x} + \boldsymbol{o}$.

Read the new coordinates off the axes

EXAMPLE

$\boldsymbol{o} = (1, 2)$, $\boldsymbol{q}_1 = (0, 1)$, $\boldsymbol{q}_2 = (-1, 0)$, and $P = (3, 3)$:

$$\boldsymbol{X} = Q^{\mathsf{T}}(\boldsymbol{x} - \boldsymbol{o}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

YOUR TURN · check it on the picture

From $O' = (1, 2)$: one step along $\boldsymbol{q}_1 = (0, 1)$, then two steps against $\boldsymbol{q}_2$, i.e. along $(1, 0)$. $(1, 2) + (0, 1) + (2, 0) = (3, 3)$. ✓

To rewrite the old line $x + y = 6$ in the new frame, substitute $x = 1 - Y$, $y = 2 + X$: it becomes $X - Y = 3$. The same line, with a new name; no point has moved.

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The same language in space

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Three columns, three coordinate directions

A 3×3 matrix stores the images of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}.$$

A quarter-turn about the z-axis, anticlockwise when seen from the tip of the z-axis.

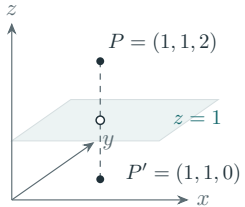
YOUR TURN · one minute

Image of $(2,1,3)$? Which points stay fixed? Are the columns orthonormal?
Answers: $(-1,2,3)$; the whole z-axis; yes, so this is an isometry.

A mirror away from the origin

Reflection in the plane $z = 1$:
$(x, y, z) \mapsto (x, y, 2 - z)$, that is

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$



3D schematic

EXAMPLE

Point $(1, 1, 2) \mapsto (1, 1, 0)$; direction $(1, 1, 2) \mapsto (1, 1, -2)$.

Every point of $z = 1$ is fixed; the columns are orthonormal, so this is an isometry, as Lecture 5 proved for every reflection. The matrix is the direction formula of Lecture 5, the translation is what the constant c adds for points.

Exit check: repair the claim

YOUR TURN · two minutes, one sentence each

1. "To transform a direction $\boldsymbol{d}$, compute $A\boldsymbol{d} + \boldsymbol{b}$."
2. "An affine map always sends a line to a line."
3. "$\det A = 1$, so the map preserves every length."
4. "To find the equation of an image, substitute the map into the old equation."

Use $A\boldsymbol{d}$: directions get no translation. A singular map may collapse a line to a point. The shear has determinant 1 and stretches $(0, 1)$ to $(1, 1)$. Substitute the *inverse* into the old equation.

Summary: three questions for every map

1. **What moves, and how?** Points: $F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$, columns of A the images of the unit arrows, $\boldsymbol{b} = F(O)$. Displacements: $A\boldsymbol{d}$. Or nothing moves and only the frame changes: $\boldsymbol{x} = \boldsymbol{o} + Q\boldsymbol{X}$.
2. **Can it be undone?** Subtract $\boldsymbol{b}$, then apply A^{-1}; in the plane that exists exactly when $ad - bc \neq 0$. Compose by substitution, undo in the reverse order.
3. **What survives?** Lines, parallelism and ratios along lines (affine combinations) always, up to collapse; areas scale by $|\det A|$; distances and angles exactly when the columns are orthonormal, and then $A^{-1} = A^{\mathsf{T}}$.

Image of a locus: substitute the inverse into the old equation. Rotation about C: $\boldsymbol{c} + R_\theta(\boldsymbol{x} - \boldsymbol{c})$.

Seminar 3, second meeting: what to prepare

Prepare: every part of Problems 3.2 and 3.3 (Core). You may be asked to present them at the board.

- Also for this meeting: Problems 3.5 and 3.6 (Medium), the board problems 3.B6–3.B8, and the Challenge 3.7 (two mirrors make a rotation) — choices for discussion, not homework to finish.

SAME SHEET AS LAST WEEK

Problem 3.1 and the board problems 3.B1–3.B5 remain available from Lecture 5.

NEXT LECTURE

Circles and ellipses: an equation, and the geometry hidden in it.