SEMINAR 1 / STUDENT PROBLEM SHEET

Vectors, projections and similarity

26 September 2026

Lectures 1 and 2 · 7 main problems and 8 board problems.

How to use this sheet

This sheet serves both meetings of Seminar 1. Each problem is marked with the lecture it needs.

- **First meeting**, after Lecture 1: Problems 1.1 and 1.4, and the board problems 1.B1–1.B5.
- **Second meeting**, after Lecture 2: Problems 1.2, 1.3, 1.5, 1.6 and 1.7, and the board problems 1.B6–1.B8.

Prepare every Core problem (1.1 for the first meeting, 1.2 and 1.3 for the second): you may be asked to present it at the board. The Medium problems are good practice, with no fixed number to prepare; the Challenge 1.7 is optional. The board problems are choices for discussion, not extra compulsory preparation: a few of them are chosen in each meeting, and volunteers are welcome. When you present, draw the configuration, name the points and vectors you use, say which fact from the lectures you apply, and check your answer. No programming or special software is needed.

Toolkit

From Lecture 1 (any frame)

- $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$; $P + \boldsymbol{v}$ is the point reached from P by the displacement $\boldsymbol{v}$; $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.
- For $A \neq B$: $\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$ is the segment $[AB]$ for $0 \leq t \leq 1$, the ray from A through B for $t \geq 0$, the line AB for $t \in \mathbb{R}$.
- The point $P \in [AB]$ with $AP : PB = m : n$ is $\boldsymbol{p} = \frac{n\boldsymbol{a} + m\boldsymbol{b}}{m+n}$; the midpoint is $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$; the centroid of ABC is $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$.
- A combination $\sum \lambda_i \boldsymbol{a}_i$ is a point if $\sum \lambda_i = 1$ and a vector if $\sum \lambda_i = 0$; in both cases it does not depend on the origin.
- If $\boldsymbol{u}, \boldsymbol{v}$ are not collinear and $\alpha\boldsymbol{u} + \beta\boldsymbol{v} = \alpha'\boldsymbol{u} + \beta'\boldsymbol{v}$, then $\alpha = \alpha'$ and $\beta = \beta'$.

From Lecture 2 (Cartesian frame)

- $\|\boldsymbol{v}\| = \sqrt{x^2 + y^2 + z^2}$, $d(A, B) = \|\boldsymbol{b} - \boldsymbol{a}\|$, $\widehat{\boldsymbol{v}} = \boldsymbol{v} / \|\boldsymbol{v}\|$ for $\boldsymbol{v} \neq \boldsymbol{0}$.
- $\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta$ for nonzero $\boldsymbol{u}, \boldsymbol{v}$; nonzero vectors are perpendicular exactly when $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.
- $\|\boldsymbol{u} \pm \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \pm 2\,\boldsymbol{u} \cdot \boldsymbol{v} + \|\boldsymbol{v}\|^2$.
- Projection onto $\boldsymbol{v} \neq \boldsymbol{0}$: $t_* = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}$, $\operatorname{proj}_{\boldsymbol{v}} \boldsymbol{u} = t_* \boldsymbol{v}$, and $\boldsymbol{u} - t_* \boldsymbol{v} \perp \boldsymbol{v}$.
- Closest point of the line $A + t\boldsymbol{v}$ to U: the foot $H = A + t_* \boldsymbol{v}$ with $t_* = \frac{\overrightarrow{AU} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}$.
- $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$; $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$; for unit vectors $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2 - 2\,\boldsymbol{x} \cdot \boldsymbol{y}$.

Problem 1.1 Core

A point moving along a segment

After Lecture 1

Let $A = (1, -1, 2)$ and $B = (5, 1, -2)$.

(a) Compute $\overrightarrow{AB}$, the midpoint M of $[AB]$, and the point P of $[AB]$ with $AP : PB = 1 : 3$.
(b) Parametrise the segment $[AB]$, the ray from A through B, and the line AB, stating the parameter range in each case.
(c) Decide whether $C = (9, 3, -6)$ and $D = (3, 1, 0)$ lie on the line AB, and whether they lie on the segment $[AB]$.
(d) What does your formula from (b) describe if B is replaced by A? Why did (b) need $A \neq B$?

Problem 1.2 Core

Is this really a right angle?

After Lecture 2

Let $\boldsymbol{u} = (1, 2, 2)$ and $\boldsymbol{v} = (2, 1, -2)$.

(a) Compute $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, $\boldsymbol{u} \cdot \boldsymbol{v}$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.
(b) Let U and V be the points with position vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Compute the distance UV directly, and check the answer with Pythagoras' theorem in the triangle OUV. (A schematic sketch is enough; do not try to draw space to scale.)
(c) Find all vectors $\boldsymbol{w}$ that are orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$, and those among them of length 3. What can you say about the three vectors $\frac{1}{3}\boldsymbol{u}$, $\frac{1}{3}\boldsymbol{v}$, $\frac{1}{3}\boldsymbol{w}$?
(d) A student writes: "$\mathbf{0} \cdot (1, -2) = 0$, so the angle between $\mathbf{0}$ and $(1, -2)$ is $90°$." What is wrong?

Problem 1.3 Core

Projection: number or vector?

After Lecture 2

Let $\boldsymbol{u} = (3, 1)$ and $\boldsymbol{v} = (1, 1)$.

(a) Find the number t for which $\boldsymbol{u} - t\boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$. Write down the projection $\mathrm{proj}_{\boldsymbol{v}}\,\boldsymbol{u}$, the remainder $\boldsymbol{r} = \boldsymbol{u} - \mathrm{proj}_{\boldsymbol{v}}\,\boldsymbol{u}$ and its length, and draw the three vectors in a Cartesian frame with equal units.
(b) Is t the projection? Is it the length of the projection? Compare t with the signed length $\boldsymbol{u} \cdot \widehat{\boldsymbol{v}}$ of the shadow.
(c) Replace $\boldsymbol{v}$ by $\boldsymbol{w} = -2\boldsymbol{v}$ and repeat (a). Which of the three answers change, and why?
(d) Now project $\boldsymbol{v}$ onto $\boldsymbol{u}$. Is $\mathrm{proj}_{\boldsymbol{u}}\,\boldsymbol{v}$ equal to $\mathrm{proj}_{\boldsymbol{v}}\,\boldsymbol{u}$? Explain with a picture.

Problem 1.4 Medium

A centroid that survives translation After Lecture 1

Let $A = (0,0)$, $B = (6,0)$ and $C = (0,3)$.

(a) Compute the centroid G with $\boldsymbol{g} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$, and check directly that G lies on the median from A, two thirds of the way from A to the midpoint of $[BC]$.

(b) Prove that in every triangle ABC the three medians pass through the point with position vector $\frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$, and that this point divides each median in the ratio $2:1$ from the vertex.

(c) Translate the triangle by a vector $\boldsymbol{w}$. Show that its centroid is translated by the same vector. How does this statement differ from the fact that G does not depend on the choice of the origin?

(d) For a tetrahedron $ABCD$ in space, let G_D be the centroid of the face ABC. Show that the point with position vector $\frac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$ lies on the segment $[DG_D]$, and find the ratio in which it divides this segment. What can you conclude about the four segments joining a vertex of the tetrahedron to the centroid of the opposite face?

Problem 1.5 Medium

Closest point, without differentiation After Lecture 2

Let U be the point with position vector $\boldsymbol{u} = (2,0,1)$ and let $\boldsymbol{v} = (1,0,1)$.

(a) Find the point H of the line $\{t\boldsymbol{v} : t \in \mathbb{R}\}$ closest to U, by writing $\|\boldsymbol{u} - t\boldsymbol{v}\|^2$ as a constant plus a nonnegative multiple of a square. What is the distance from U to the line?

(b) Prove in general: if $\boldsymbol{v} \neq \boldsymbol{0}$ and $t_* = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}$, then

$$\|\boldsymbol{u} - t\boldsymbol{v}\|^2 = \|\boldsymbol{u} - t_*\boldsymbol{v}\|^2 + (t - t_*)^2\,\|\boldsymbol{v}\|^2 \qquad \text{for all } t \in \mathbb{R}.$$

Deduce that $t_*\boldsymbol{v}$ is the unique point of the line closest to U, and that UH is perpendicular to the line.

(c) Find the point of the line through $A = (1,1,1)$ with direction $(1,0,1)$ closest to $P = (3,2,5)$, and the distance from P to this line.

Problem 1.6 Medium

Two meanings of similarity After Lecture 2

A toy search engine represents a text by the numbers of occurrences of the words *geometry*, *vector* and *music*, in this order. The query "geometry vector" is $\boldsymbol{q} = (1,1,0)$, and three documents are represented by

$$\boldsymbol{a} = (4,4,2), \qquad \boldsymbol{b} = (1,0,1), \qquad \boldsymbol{c} = (0,0,3).$$

(a) Rank the three documents by their Euclidean distance from $\boldsymbol{q}$.

(b) Rank them by cosine similarity with $\boldsymbol{q}$.

(c) Normalise all four vectors and rank the documents by the distance between unit vectors. Prove that $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2 - 2\,\boldsymbol{x}\cdot\boldsymbol{y}$ for unit vectors $\boldsymbol{x}, \boldsymbol{y}$, and explain why the rankings in (b) and (c) always agree.

(d) Add the document $2\boldsymbol{a}$, which is $\boldsymbol{a}$ with every word repeated. Where does it appear in each ranking? Which ranking answers the question "which document is about the same subject as the query?", and what information does it ignore?

Problem 1.7 Challenge

An equilateral triangle on an integer grid? After Lectures 1 and 2

(a) Prove that no equilateral triangle in the plane has all three vertices in $\mathbb{Z}^2$, that is, at points with integer coordinates.
(b) Find three points of $\mathbb{Z}^3$ that are the vertices of an equilateral triangle. Why does this not contradict (a)?
(c) Find four points of $\mathbb{Z}^3$ that are the vertices of a regular tetrahedron (all six edges of equal length).

Board problems

These problems are for discussion at the board. They are optional: nobody is expected to prepare all of them, and in each meeting a few are chosen. Volunteers are welcome. Work in a Cartesian frame, draw the configuration first, and say which facts from the lectures you use.

After Lecture 1

Problem 1.B1 Board

Three points, three parallelograms After Lecture 1

The points $A = (1, 1)$, $B = (5, 1)$ and $C = (2, 4)$ are three vertices of a parallelogram, but we do not know which of them are adjacent.

(a) Find all possible fourth vertices, and for each one list the vertices of the parallelogram in cyclic order. Draw the three parallelograms.
(b) Justify each answer with vectors, and check that in each parallelogram the two diagonals have the same midpoint.
(c) Show that A, B and C are the midpoints of the sides of the triangle formed by the three fourth vertices.
(d) What goes wrong if the three given points are collinear?

Problem 1.B2 Board

Rebuild a triangle from its midpoints After Lecture 1

A triangle ABC has disappeared from a drawing; only the midpoints of its sides are left:

$$M = (2, 1) \text{ on } [AB], \qquad N = (4, 3) \text{ on } [BC], \qquad P = (1, 4) \text{ on } [CA].$$

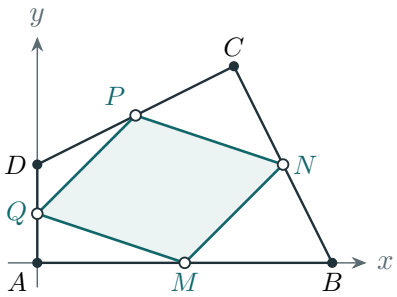
(a) Recover A, B and C, and check your answer.
(b) Express $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ in terms of $\boldsymbol{m}$, $\boldsymbol{n}$, $\boldsymbol{p}$, and explain why there is only one such triangle. How is this related to 1.B1?
(c) Now M, N, P, Q are given as the midpoints of the sides $[AB]$, $[BC]$, $[CD]$, $[DA]$ of an unknown quadrilateral $ABCD$. Show that a solution can exist only if $\boldsymbol{m} + \boldsymbol{p} = \boldsymbol{n} + \boldsymbol{q}$, and that in that case there are infinitely many quadrilaterals with these midpoints.

Problem 1.B3 Board

Midpoints tame a quadrilateral

After Lecture 1

Let $A = (0,0)$, $B = (6,0)$, $C = (4,4)$, $D = (0,2)$, in this cyclic order, and let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$, $[DA]$.



(a) Compute M, N, P, Q and show with vectors that $MNPQ$ is a parallelogram. (The picture suggests it; it does not prove it.)

(b) Prove that for *any* four points A, B, C, D,

$$\overrightarrow{MN} = \overrightarrow{QP} = \tfrac{1}{2}\,\overrightarrow{AC} \qquad \text{and} \qquad \overrightarrow{NP} = \overrightarrow{MQ} = \tfrac{1}{2}\,\overrightarrow{BD}.$$

(c) Show that the diagonals of $MNPQ$ meet at the point with position vector $\tfrac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$, and compute it for the data above.

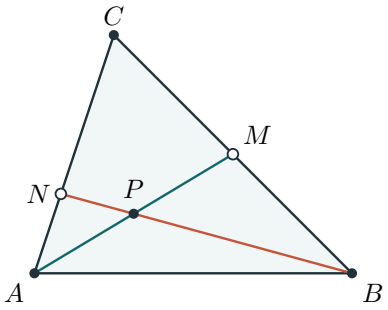
(d) The proof of (b) never used that A, B, C, D lie in a plane. What does it say about four points of space that are not coplanar, for instance $A = (0,0,0)$, $B = (2,0,0)$, $C = (2,2,0)$, $D = (0,0,2)$?

Problem 1.B4 Board

Where two cevians meet

After Lecture 1

In a triangle ABC, let M be the midpoint of $[BC]$ and let N be the point of $[AC]$ with $AN = \tfrac{1}{3}AC$. The segments $[AM]$ and $[BN]$ meet at a point P.



(a) Put $\boldsymbol{b} = \overrightarrow{AB}$ and $\boldsymbol{c} = \overrightarrow{AC}$. Write $\overrightarrow{AP}$ in two ways, as $s\,\overrightarrow{AM}$ and as $\overrightarrow{AB} + r\,\overrightarrow{BN}$ for some numbers s, r, and express both in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

(b) Why are $\boldsymbol{b}$ and $\boldsymbol{c}$ not collinear? Compare coefficients to find s and r. In which ratios does P divide $[AM]$ and $[BN]$? Does the picture agree?

(c) Check your answer for $A = (0,0)$, $B = (6,0)$, $C = (0,6)$ by computing P in coordinates.

(d) Repeat (b) with $AN = \lambda\, AC$ for a number $0 < \lambda < 1$. For which λ is P the centroid of the triangle?

Problem 1.B5 Board

Midpoint flips and a jumping frog

After Lecture 1

The *point reflection* (or midpoint flip) in a point M sends a point X to the point X' for which M is the midpoint of $[XX']$.

(a) Show that $\boldsymbol{x'} = 2\boldsymbol{m} - \boldsymbol{x}$. For $M = (1,1)$ and $N = (4,2)$, reflect $X = (-1,3)$ in M to get Y, and then reflect Y in N to get Z.

(b) Prove that for *every* point X, reflecting first in M and then in N moves X by the vector $2\,\overrightarrow{MN}$. So two point reflections make a translation. Does the order of the two reflections matter?

(c) A frog sits at a point X_0 and jumps repeatedly: first to its reflection in a point A, then to the reflection of its new position in B, then in C, then again in A, B, C, and so on. Show that after three jumps it is at the reflection of X_0 in the point Q with $\boldsymbol{q} = \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$, and that after six jumps it is back at X_0. Check with $A = (0,0)$, $B = (2,0)$, $C = (1,2)$, $X_0 = (1,-1)$.

After Lecture 2

Problem 1.B6 Board

The parallelogram law and the medians

After Lecture 2

Let $ABCD$ be a parallelogram, with $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{AD}$.

(a) Express the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$, and prove the *parallelogram law*

$$AC^2 + BD^2 = 2\,AB^2 + 2\,AD^2 :$$

the squares of the two diagonals add up to the squares of the four sides.

(b) Prove that $(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}) = \|\boldsymbol{u}\|^2 - \|\boldsymbol{v}\|^2$. Deduce that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.

(c) In a triangle ABC with side lengths $a = BC$, $b = CA$, $c = AB$, let m_a be the length of the median from A. Use (a) to prove that $m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$, and check it for $A = (2,4)$, $B = (0,0)$, $C = (6,0)$.

(d) Let O be the midpoint of $[AB]$, and let C be a point with $OC = OA$. Use (b) to show that $\overrightarrow{CA}\cdot\overrightarrow{CB} = 0$. Which classical theorem is this?

Problem 1.B7 Board

The altitudes of a triangle meet

After Lecture 2

An *altitude* of a triangle is the line through a vertex perpendicular to the opposite side.

(a) Prove that for all vectors $\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$,

$$(\boldsymbol{x}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{c}) + (\boldsymbol{x}-\boldsymbol{b})\cdot(\boldsymbol{c}-\boldsymbol{a}) + (\boldsymbol{x}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{b}) = 0.$$

(b) In a triangle ABC, let H be the point where the altitudes from A and from B meet. Use (a) to show that CH is perpendicular to AB. Conclude that the three altitudes of a triangle pass through one point, the *orthocentre*.

(c) Find the orthocentre of the triangle $A=(0,0)$, $B=(6,0)$, $C=(2,4)$, and check the three perpendicularity conditions.

(d) Where is the orthocentre of a right triangle?

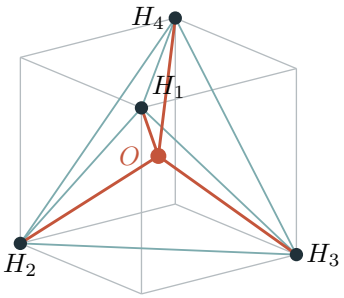
Problem 1.B8 Board

The methane angle

After Lecture 2

In a methane molecule the carbon atom sits at the centre of a regular tetrahedron whose vertices are the four hydrogen atoms. A convenient model puts the carbon atom at $O=(0,0,0)$ and the hydrogen atoms at four alternate vertices of the cube $[-1,1]^3$:

$$H_1=(1,1,1),\quad H_2=(1,-1,-1),\quad H_3=(-1,1,-1),\quad H_4=(-1,-1,1).$$



(a) Show that the six distances between hydrogen atoms are equal, and that O is at the same distance from all four.

(b) Compute the bond angle $\angle H_1OH_2$, to two decimals of a degree. Why are all six bond angles equal?

(c) Show that $\overrightarrow{OH_1}+\overrightarrow{OH_2}+\overrightarrow{OH_3}+\overrightarrow{OH_4}=\mathbf{0}$. What is O for the four points H_i, in the language of Lecture 1?

(d) Forget the coordinates. If $\boldsymbol{u}_1,\dots,\boldsymbol{u}_4$ are unit vectors with $\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3+\boldsymbol{u}_4=\mathbf{0}$ and $\boldsymbol{u}_i\cdot\boldsymbol{u}_j=c$ for all $i\neq j$, show that $c=-\frac{1}{3}$.

(e) The angle between two lines is the smaller of the angles they form. Find the angle between two space diagonals of the cube. Why is it not the bond angle?