SEMINAR 2 / STUDENT PROBLEM SHEET

Lines, planes and spatial reasoning

9 October 2026

Lectures 3 and 4 · 7 main problems and 8 board problems.

How to use this sheet

This sheet serves both meetings of Seminar 2. Each problem is marked with the lecture it needs.

- **First meeting**, after Lecture 3: Problems 2.1 and 2.4, and the board problems 2.B1–2.B5. Four of the board problems (2.B1–2.B4) practise the part of Lecture 2 that was completed at the start of Lecture 3: the closest point of a line, the Cauchy–Schwarz and triangle inequalities, and cosine similarity.
- **Second meeting**, after Lecture 4: Problems 2.2, 2.3, 2.5 and 2.6, and the board problems 2.B6–2.B8. The Challenge 2.7 uses both lectures.

Prepare every Core problem (2.1 for the first meeting, 2.2 and 2.3 for the second): you may be asked to present it at the board. The Medium problems are good practice, with no fixed number to prepare; the Challenge 2.7 is optional. The board problems are choices for discussion, not extra compulsory preparation: a few of them are chosen in each meeting, and volunteers are welcome. When you present, draw the configuration (space pictures are schematic, say so), name the points and vectors you use, say which fact from the lectures you apply, and check your answer in the original equations. All coordinates are taken in a Cartesian frame with a common unit, right-handed in space. No programming or special software is needed.

Toolkit

From Lecture 3 (with the end of Lecture 2)

- Closest point of the line $A + t\boldsymbol{v}$ ($\boldsymbol{v} \neq \boldsymbol{0}$) to U: project $\overrightarrow{AU}$, not the position vector of U: $t_* = \frac{\overrightarrow{AU}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}$, $H = A + t_*\boldsymbol{v}$, and for every $X = A + t\boldsymbol{v}$, $UX^2 = UH^2 + (t - t_*)^2 \|\boldsymbol{v}\|^2$. On a ray or segment, take the admissible t closest to t_*.
- $|\boldsymbol{u}\cdot\boldsymbol{v}| \leq \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$, with equality exactly for collinear vectors; $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, with equality (nonzero vectors) exactly for the same direction *and* sense. For nonzero vectors $\operatorname{cosim}(\boldsymbol{u},\boldsymbol{v}) = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \in [-1,1]$, and for unit vectors $\|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2 - 2\,\boldsymbol{x}\cdot\boldsymbol{y}$.
- A line is $A + t\boldsymbol{v}$ with $\boldsymbol{v} \neq \boldsymbol{0}$ and $t \in \mathbb{R}$; $t \geq 0$ gives the ray from A, and $A + t\,\overrightarrow{AB}$ with $0 \leq t \leq 1$ the segment $[AB]$. The same t in every coordinate. $B + s\boldsymbol{w}$ is the same line exactly when B is on it and $\boldsymbol{w} = c\boldsymbol{v}$, $c \neq 0$.
- In the plane, the line through (a,b) with direction (p,q) is $q(x-a) - p(y-b) = 0$; the line through A with normal $\boldsymbol{n} = (\alpha,\beta) \neq \boldsymbol{0}$ is $\boldsymbol{n}\cdot\overrightarrow{AX} = 0$, i.e. $\alpha x + \beta y = \alpha a + \beta b$, with direction $(-\beta,\alpha)$. Vertical lines need no slope.
- Two lines $A + t\boldsymbol{u}$, $B + s\boldsymbol{v}$: if $\boldsymbol{v} = c\boldsymbol{u}$, test whether B is on the first line (same line, or distinct parallel lines); otherwise solve $A + t\boldsymbol{u} = B + s\boldsymbol{v}$ with *independent* t, s and check every coordinate (one common point, or skew lines in space). Angle between lines:

$\cos\varphi = \frac{|\boldsymbol{u}\cdot\boldsymbol{v}|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$, $\varphi \in [0, \frac{\pi}{2}]$; perpendicular lines must also meet.

- Rays and segments: intersect the lines, then check the parameter domains. Two moving points collide only if they are at the same place at the same *time*: one shared parameter, and only when the clock is shared.

From Lecture 4 (right-handed Cartesian frame)

- $\boldsymbol{u} \times \boldsymbol{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$: perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$, of length $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\sin\theta$ (the area of their parallelogram), right-handed; $\boldsymbol{v}\times\boldsymbol{u} = -\boldsymbol{u}\times\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{0}$ exactly for collinear vectors. $\text{area}(ABC) = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$.
- The plane through A with normal $\boldsymbol{n} = (a,b,c) \neq \boldsymbol{0}$: $\boldsymbol{n}\cdot\overrightarrow{AX} = 0$, i.e. $ax+by+cz = k$ with $k = \boldsymbol{n}\cdot\boldsymbol{a}$; the coefficients of an equation are a normal, and the equation may be multiplied by any nonzero number. Through three non-collinear points: normal $\overrightarrow{AB}\times\overrightarrow{AC}$. With two non-collinear directions: $A + s\boldsymbol{u} + r\boldsymbol{v}$, $s, r \in \mathbb{R}$. A missing variable is free, not zero.
- A line $P + t\boldsymbol{d}$ against a plane $\boldsymbol{n}\cdot\boldsymbol{x} = k$: substitute and read $(\boldsymbol{n}\cdot\boldsymbol{d})\,t = k - \boldsymbol{n}\cdot\boldsymbol{p}$ — one point, no point ($\boldsymbol{n}\cdot\boldsymbol{d} = 0$, $P \notin \Pi$), or the whole line ($\boldsymbol{n}\cdot\boldsymbol{d} = 0$, $P \in \Pi$); then check the domain of t for a ray or segment.
- Two planes: parallel normals give the same plane or parallel planes (compare the constants); otherwise a line with direction $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ through one common point.
- Angles, always in $[0, \frac{\pi}{2}]$: between planes $\cos\alpha = \frac{|\boldsymbol{n}_1\cdot\boldsymbol{n}_2|}{\|\boldsymbol{n}_1\|\|\boldsymbol{n}_2\|}$; between a line and a plane $\sin\alpha = \frac{|\boldsymbol{d}\cdot\boldsymbol{n}|}{\|\boldsymbol{d}\|\|\boldsymbol{n}\|}$.
- $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$ is a signed volume: the parallelepiped has volume $|[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]|$, the tetrahedron $ABCD$ has volume $\frac{1}{6}|[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]|$, and the four points are coplanar exactly when the triple product is 0.
- A ray hits the triangular face ABC when it hits the plane with $t \geq 0$ at a point H with $\overrightarrow{AH} = s\,\overrightarrow{AB} + r\,\overrightarrow{AC}$, $s \geq 0$, $r \geq 0$, $s + r \leq 1$.

Problem 2.1 Core

One line, three descriptions

After Lecture 3

In the Cartesian plane let $A = (2, -1)$ and $B = (5, 1)$.

(a) Compute $\overrightarrow{AB}$ and write the line AB as $A + t\,\overrightarrow{AB}$, with the domain of t. Find a nonzero normal vector of the line and a Cartesian equation $\alpha x + \beta y = \gamma$; check that A and B satisfy it.

(b) Decide whether $P = (-4, -5)$, $Q = (\frac{7}{2}, 0)$ and $R = (-1, -2)$ lie on the line AB, on the ray from A through B, and on the segment $[AB]$. Find the parameter of each point first (if it has one), then decide.

(c) Explain why $B + s(-6, -4)$, $s \in \mathbb{R}$, describes the same line. How are s and t related? Do $s \geq 0$ and $t \geq 0$ describe the same ray? What do the two rays have in common?

(d) Repeat (a) for the line through $C = (-3, 2)$ and $D = (-3, 5)$: vector equation, normal vector, Cartesian equation. Why would a slope formula have failed here, and why did the method of (a) not?

Problem 2.2 Core

A triangle gives a normal and an area

After Lecture 4

In a right-handed Cartesian frame let $A = (1, 0, 0)$, $B = (0, 2, 0)$ and $C = (0, 0, 3)$ be the vertices of a triangular face of a 3D model.

(a) Compute $\boldsymbol{u} = \overrightarrow{AB}$, $\boldsymbol{v} = \overrightarrow{AC}$ and $\boldsymbol{n} = \boldsymbol{u} \times \boldsymbol{v}$, and check that $\boldsymbol{n} \cdot \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{v} = 0$.

(b) Find the area of the triangle ABC and a Cartesian equation of its plane. Check that all three vertices satisfy the equation.

(c) Swap the roles of B and C: what happens to the normal, to the plane equation and to the area? Then replace C by $C' = (-1, 4, 0)$: does the construction still produce a plane? How are A, B, C' situated?

(d) The line through the origin with direction $\boldsymbol{n}$ meets the plane of ABC in a point H. Find H and the length OH. (In Lecture 5, OH will turn out to be the distance from O to the plane.)

Problem 2.3 Core

Does the ray actually hit the plane?

After Lecture 4

Let Π be the plane $x + 2y + 2z = 6$ and let $P = (2, 2, 2)$. A ray from P has the form $P + t\boldsymbol{d}$ with $t \geq 0$.

(a) Take $\boldsymbol{d} = (0, -1, -1)$. Substitute the ray into the equation of Π, solve for t, and find the point H where the ray hits the plane. Check that H satisfies the equation and that the parameter is admissible.

(b) Replace $\boldsymbol{d}$ by $-\boldsymbol{d}$. Where does the *line* through P with this direction meet Π? Does the new ray reach that point?

(c) Take $\boldsymbol{e} = (2, -1, 0)$. What does the substitution give, and what does it mean geometrically? Answer the same question for the ray from $Q = (2, 1, 1)$ with direction $\boldsymbol{e}$.

(d) Does the segment from P to $R = (2, -1, -1)$ cross the plane? And the segment from P to $S = (2, \frac{3}{2}, \frac{3}{2})$? Give the parameter in each case.

Problem 2.4 Medium

Perpendicular directions, but no intersection

After Lecture 3

In space consider the lines

$$\ell:\ X(t) = (t,\, t,\, 0), \qquad m:\ Y(s) = (1+s,\, 1-s,\, 2), \qquad t, s \in \mathbb{R}.$$

(a) Show that the directions of ℓ and m are perpendicular. Then decide, using all three coordinates, whether the lines meet. Are they parallel, intersecting or skew? Draw a labelled schematic picture.

(b) Expand $\|Y(s) - X(t)\|^2$ and find the smallest distance between a point of ℓ and a point of m, without calculus. Which two points realise it? Check that the segment joining them is perpendicular to both lines, and write down the line through these two points.

(c) Replace m by $m_h:\ Y(s) = (1+s,\, 1-s,\, h)$, where $h \in \mathbb{R}$. Classify the pair ℓ, m_h for every value of h and give the smallest distance between them. For which h is it correct to call ℓ and m_h *perpendicular lines*?

Problem 2.5 Medium

A line hidden inside two planes

After Lecture 4

Consider the planes $\Pi_1:\ x + y + z = 2$ and $\Pi_2:\ x - y + z = 0$.

(a) Subtract the two equations. Then use one coordinate as a free parameter to describe every point of $\Pi_1 \cap \Pi_2$ in the form $A + t\boldsymbol{d}$.

(b) Compute the cross product of the two normals and compare it with your direction $\boldsymbol{d}$. Verify that your line satisfies both plane equations for *every* t.

(c) Find the angle between Π_1 and Π_2. Then compare Π_1 with $\Pi_k:\ 2x + 2y + 2z = k$: for which k do the two planes coincide, and what are they otherwise? What does the cross product of their normals tell you, and what does it not tell you?

(d) Find the point of the line $\Pi_1 \cap \Pi_2$ closest to the origin, and its distance from the origin. (Lecture 3, the closest point of a line.)

Problem 2.6 Medium

Tetrahedra and coplanarity

After Lecture 4

Let $A = (0,0,0)$, $B = (2,0,0)$, $C = (0,3,0)$ and $D_h = (1,1,h)$, where $h \in \mathbb{R}$.

(a) Compute the scalar triple product $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD_h}] = (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD_h}$ and the volume of the tetrahedron $ABCD_h$. Explain the factor $\frac{1}{6}$ with a base area and a height.

(b) For which h are the four points coplanar? Compare $h = 2$ with $h = -2$: what changes, and what stays the same?

(c) Move the apex to $D_h' = D_h + \alpha\,\overrightarrow{AB} + \beta\,\overrightarrow{AC}$ with $\alpha, \beta \in \mathbb{R}$. Show that the volume does not change, and explain this geometrically.

(d) Without computing a new cross product, find the volume of the tetrahedron with vertices $(1,1,1)$, $(3,1,1)$, $(1,4,1)$ and $(2,2,3)$.

Problem 2.7 Challenge

A triangle, a point and three weights

After Lectures 3 and 4

Let A, B, C be non-collinear points in space, with position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in a fixed frame. A point P with

$$\boldsymbol{p} = \lambda \boldsymbol{a} + \mu \boldsymbol{b} + \nu \boldsymbol{c}, \qquad \lambda + \mu + \nu = 1,$$

is said to have *barycentric weights* (λ, μ, ν) with respect to the triangle ABC. (Lecture 1: a combination of points whose coefficients add up to 1 is a point, and does not depend on the origin.)

(a) Show that the condition is equivalent to $\overrightarrow{AP} = \mu\,\overrightarrow{AB} + \nu\,\overrightarrow{AC}$. Deduce that every point of the plane ABC has exactly one triple of weights, and that no point outside this plane has any.

(b) Prove that P lies in the closed triangle ABC (interior and boundary) exactly when $\lambda, \mu, \nu \geq 0$. For one direction, regard the triangle as the union of the segments from A to the points of $[BC]$; for the other, group the terms in $\mu\boldsymbol{b} + \nu\boldsymbol{c}$. Treat $\mu + \nu = 0$ separately.

(c) Let $A = (0,0,0)$, $B = (2,0,0)$, $C = (0,2,0)$. Find the weights of $P = (\frac{1}{2}, \frac{1}{2}, 0)$ and of $Q = (2,1,0)$, and decide which of them lies in the triangle. What do the signs of the weights say about the position of Q?

(d) A ray starts at $R = (\frac{1}{2}, \frac{1}{2}, 2)$ with direction $(0,0,-1)$. Does it hit the face ABC? Answer the same question for the ray from $R' = (2,1,2)$ with the same direction. Explain why hitting the plane of a face is only the first test: what else has to be checked?

Board problems

These problems are for discussion at the board. They are optional: nobody is expected to prepare all of them, and in each meeting a few are chosen. Volunteers are welcome. Work in a Cartesian frame, draw the configuration first, and say which facts from the lectures you use.

After Lecture 3

Problem 2.B1 Board

A closest point may lie beyond the segment After Lecture 3

In the Cartesian plane let $A = (1, 0)$, $B = (3, 1)$ and $U = (4, 4)$, and let $\boldsymbol{d} = \overrightarrow{AB}$.

(a) Compute $t_* = \dfrac{\overrightarrow{AU} \cdot \boldsymbol{d}}{\|\boldsymbol{d}\|^2}$ and the foot $H = A + t_*\boldsymbol{d}$. Check that $\overrightarrow{HU} \perp \boldsymbol{d}$, and draw A, B, H, U with equal units on the axes.

(b) For $X(t) = A + t\boldsymbol{d}$, prove the identity

$$\left\|\overrightarrow{X(t)U}\right\|^2 = \left\|\overrightarrow{HU}\right\|^2 + (t - t_*)^2 \, \|\boldsymbol{d}\|^2$$

and explain why the cross term disappears. Which point of the whole line is closest to U, and why is it the only one?

(c) Now restrict t to the ray $t \geq 0$, and then to the segment $0 \leq t \leq 1$. Find the closest point to U in each case, with the distance. Why is the foot of the perpendicular not always the closest point of a segment?

Problem 2.B2 Board

A sharp bound and the direction that attains it After Lecture 3

(a) Let $\boldsymbol{v} \neq \boldsymbol{0}$ and let $\boldsymbol{u}$ be any vector. Expand the squared norm in

$$0 \leq \left\|\boldsymbol{u} - \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\,\boldsymbol{v}\right\|^2$$

and deduce the Cauchy–Schwarz inequality $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$. When exactly does equality hold? What about $\boldsymbol{v} = \boldsymbol{0}$?

(b) Find the largest and the smallest value of $3x + 4y$ over all (x, y) with $x^2 + y^2 = 1$, and every point where they are attained. Describe the maximiser geometrically.

(c) What changes if the constraint is $x^2 + y^2 \leq 1$ instead? Explain why an upper bound alone, without a point attaining it, does not determine the maximum.

Problem 2.B3 Board

When does a detour cost nothing? After Lecture 3

Let $A = (0, 0)$ and $B = (4, 0)$, and let C be any point of the plane.

(a) Use $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$ and the triangle inequality to show that the broken route $A \to C \to B$ has length at least 4. Say exactly where C must lie for the length to be exactly 4, including the cases $C = A$ and $C = B$.

(b) Compute the route lengths for $C_1 = (2, 0)$, $C_2 = (2, 3)$ and $C_3 = (6, 0)$. Is it enough for C to be on the line AB?

(c) Derive the triangle inequality from Cauchy–Schwarz by expanding $\|\boldsymbol{u} + \boldsymbol{v}\|^2$. For nonzero $\boldsymbol{u}, \boldsymbol{v}$, why does equality require the same *sense* and not merely parallel directions?

Problem 2.B4 Board

Close in size, or close in direction? After Lecture 3

Two features of an object are recorded as two nonnegative numbers in the same units. For a query $\boldsymbol{q} = (1, 0)$, compare the records $\boldsymbol{a} = (4, 0)$ and $\boldsymbol{b} = (1, 1)$.

(a) Rank $\boldsymbol{a}$ and $\boldsymbol{b}$ by their Euclidean distance from $\boldsymbol{q}$, and then by their cosine similarity $\operatorname{cosim}(\boldsymbol{q}, \boldsymbol{x}) = \dfrac{\boldsymbol{q} \cdot \boldsymbol{x}}{\|\boldsymbol{q}\| \, \|\boldsymbol{x}\|}$ with $\boldsymbol{q}$. Explain in words what each ranking measures.

(b) Normalise the three vectors. Prove that for any unit vectors $\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{v}}$

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{v}}\|^2 = 2 - 2\,\widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{v}},$$

and use it to compare the distances between the normalised vectors with your cosine ranking.

(c) A missing record is stored as $\boldsymbol{c} = (0, 0)$. Which of the two comparisons is still defined for $\boldsymbol{c}$? Can $\boldsymbol{c}$ be normalised? What happens to the cosine similarity if $\boldsymbol{a}$ is replaced by $-\boldsymbol{a}$? (Not a valid record any more — this only tests the formula.)

Problem 2.B5 Board

The paths cross: do the travellers meet? After Lecture 3

Two walkers move in the plane; the time $t \geq 0$ is measured on a common clock, in the same units for both:

$$P(t) = (t,\, t), \qquad Q(t) = (4,\, 0) + t\,(0, 2).$$

(a) Forget the clock for a moment. Find the point where the two *paths* cross by solving $P(t) = Q(s)$ with independent parameters $t, s \geq 0$.

(b) Now use the clock. Do the walkers collide? Explain why the equations of (a) and of (b) answer different questions.

(c) Keep both velocities but start the second walker at $(4, -4)$. Is there a collision now? Give the time and the place. Finally, explain why a general path-crossing problem needs a separate look at parallel and coincident paths.

After Lecture 4

Problem 2.B6 Board

Which equations really describe a plane? After Lecture 4

In space consider the equations

$$\Pi:\ x - 2z = 3, \qquad E_1:\ 2x - 4z = 6, \qquad E_2:\ 0x + 0y + 0z = 0, \qquad E_3:\ 0x + 0y + 0z = 5.$$

(a) Give a point of Π, a nonzero normal, and a description $A + s\boldsymbol{u} + r\boldsymbol{v}$ with two parameters. Check that $\boldsymbol{u}$ and $\boldsymbol{v}$ are not collinear and that both are perpendicular to your normal.

(b) Describe the set of points satisfying each of E_1, E_2, E_3. Which of them is the plane Π again? Why does a Cartesian equation of a plane need a nonzero normal?

(c) A student says: "There is no y in the equation of Π, so $y = 0$ on Π." Correct the statement. Does Π contain every line parallel to the y-axis, or only some of them? Which?

Problem 2.B7 Board

An angle that cannot depend on the sign of a normal After Lecture 4

Let $\Pi: \; z = 0$ and $\Sigma: \; x + z = 2$. The angle between two planes means their smaller angle, in $[0, \frac{\pi}{2}]$.

(a) Find a normal of each plane and the angle between the planes, using $\cos\alpha = \dfrac{|\boldsymbol{n}_1 \cdot \boldsymbol{n}_2|}{\|\boldsymbol{n}_1\| \, \|\boldsymbol{n}_2\|}$. Why is the absolute value needed? Test your explanation by reversing one of the normals.

(b) Find the line $\Pi \cap \Sigma$ and check that its direction is perpendicular to both normals. Draw a labelled schematic picture of the two planes and their intersection; do not read the angle off the drawing.

(c) Replace Σ first by $x = 2$ and then by $z = 5$. In each case, what is the angle between the planes, and what is their intersection?

Problem 2.B8 Board

Sliding an edge without changing the volume After Lecture 4

Three edges of a parallelepiped meet at a vertex; in a right-handed Cartesian frame they are

$$\boldsymbol{u} = (1, 0, 1), \qquad \boldsymbol{v} = (0, 2, 0), \qquad \boldsymbol{w} = (0, 0, 3).$$

(a) Compute the scalar triple product $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$, the volume of the parallelepiped, and the volume of the tetrahedron with vertices at the common vertex and the three endpoints of the edges.

(b) Replace $\boldsymbol{w}$ by $\boldsymbol{w}' = \boldsymbol{w} + 2\boldsymbol{u} - \boldsymbol{v}$. Compute the new triple product; then explain, without any new coordinate calculation, why it had to be the same as before.

(c) Replace $\boldsymbol{w}$ instead by $\boldsymbol{w}'' = \boldsymbol{u} + \boldsymbol{v}$. All three edges are nonzero, and no two are parallel. Why is the volume nevertheless 0? What has the parallelepiped collapsed into?