SEMINAR 3 / STUDENT PROBLEM SHEET

Distances, mirrors and affine maps

22 October 2026

Lectures 5 and 6 · 7 main problems and 8 board problems.

How to use this sheet

This sheet serves both meetings of Seminar 3. Each problem is marked with the lecture it needs.

- **First meeting**, after Lecture 5: Problems 3.1 and 3.4, and the board problems 3.B1–3.B5 (two cushions, a plane in space, a point and a direction in a mirror, a margin to a boundary, and the cases in which the shortest-route argument changes). No matrices are needed.
- **Second meeting**, after Lecture 6: Problems 3.2, 3.3, 3.5 and 3.6, and the board problems 3.B6–3.B8 (a passive frame, a collapsing map, and why affine maps keep midpoints). The Challenge 3.7 uses both lectures.

Prepare every Core problem (3.1 for the first meeting, 3.2 and 3.3 for the second): you may be asked to present it at the board. The Medium problems are good practice, with no fixed number to prepare; the Challenge 3.7 is optional. The board problems are choices for discussion, not extra compulsory preparation: a few of them are chosen in each meeting, and volunteers are welcome. When you present, draw the configuration on equal-scale axes (space pictures are schematic, say so), say which objects are points, which are vectors and which are numbers, name the fact from the lectures you apply, and check your answer in the original equations. All coordinates are taken in an orthonormal Cartesian frame with a common unit, right-handed in space. No programming or special software is needed.

Toolkit

From Lecture 5

- A line in the plane and a plane in space have the same kind of equation, $E:\ \boldsymbol{n}\cdot\boldsymbol{x}=c$ with $\boldsymbol{n}\neq\boldsymbol{0}$; every displacement inside E is perpendicular to $\boldsymbol{n}$. The *residual* of a point P is $\boldsymbol{n}\cdot\boldsymbol{p}-c$; it is 0 exactly when $P\in E$, and it is not a distance.
- Foot of the perpendicular from P: $H=P-\dfrac{\boldsymbol{n}\cdot\boldsymbol{p}-c}{\|\boldsymbol{n}\|^2}\,\boldsymbol{n}$. Check it three ways: $H\in E$, $\overrightarrow{HP}\parallel\boldsymbol{n}$, and $PH=d(P,E)=\dfrac{|\boldsymbol{n}\cdot\boldsymbol{p}-c|}{\|\boldsymbol{n}\|}$. For a line given as $A+t\boldsymbol{d}$, project instead: $t_*=\frac{\overrightarrow{AP}\cdot\boldsymbol{d}}{\boldsymbol{d}\cdot\boldsymbol{d}}$, $H=A+t_*\boldsymbol{d}$ (the only route for a line in space).
- $PX^2=PH^2+HX^2$ for every $X\in E$: this is why H is the *unique* closest point. Checking a few points is not a proof; the identity is.
- Signed distance $s(P)=\dfrac{\boldsymbol{n}\cdot\boldsymbol{p}-c}{\|\boldsymbol{n}\|}$: positive on the side $\boldsymbol{n}$ points to, negative on the other, 0 on E; $\overrightarrow{HP}=s(P)\,\hat{\boldsymbol{n}}$ with $\hat{\boldsymbol{n}}=\boldsymbol{n}/\|\boldsymbol{n}\|$. Multiplying the whole equation by $\lambda\neq0$ changes neither E, nor H, nor the distance; for $\lambda<0$ it changes the sign of s. Parallel lines or planes $\boldsymbol{n}\cdot\boldsymbol{x}=c_1$, $\boldsymbol{n}\cdot\boldsymbol{x}=c_2$ (same $\boldsymbol{n}$!) are $|c_1-c_2|/\|\boldsymbol{n}\|$ apart.

- Mirror image of a point: $P^* = 2H - P = P - 2\dfrac{\boldsymbol{n}\cdot\boldsymbol{p} - c}{\|\boldsymbol{n}\|^2}\boldsymbol{n}$ (the foot is the midpoint). Mirror image of a direction: $\boldsymbol{v}^* = \boldsymbol{v} - 2\dfrac{\boldsymbol{v}\cdot\boldsymbol{n}}{\|\boldsymbol{n}\|^2}\boldsymbol{n}$ (no c: keep the part along the mirror, reverse the part across it). The line $P + t\boldsymbol{v}$ goes to $P^* + t\boldsymbol{v}^*$. Reflections preserve distances and fix the points of the mirror.
- Shortest route $A \to M \to B$ with M on a line ℓ, A and B strictly on the same side: reflect A to A^*; then $AM + MB = A^*M + MB \geq A^*B$ (triangle inequality), with equality exactly when $M \in [A^*B]$, and that segment crosses ℓ at exactly one point. Three steps every time: a bound, its equality case, an allowed point attaining it. A bound that no allowed point attains is not the minimum.

From Lecture 6 (a tuple is a column when a matrix acts on it)

- The columns of a matrix are the images of the coordinate arrows: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$, and likewise with three columns in space. The action is linear and sends $\mathbf{0}$ to $\mathbf{0}$.
- An affine map is $F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{b} = F(O)$ (an active map: points move, the frame stays). Points receive the translation, displacements do not: $\overrightarrow{F(P)F(Q)} = A\overrightarrow{PQ}$, and $P + t\boldsymbol{d}$ goes to $F(P) + tA\boldsymbol{d}$, a line unless $A\boldsymbol{d} = \mathbf{0}$. Affine combinations are preserved: midpoints, centroids, division ratios.
- Composition by substitution: if $F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ and $G(\boldsymbol{x}) = C\boldsymbol{x} + \boldsymbol{d}$, then $(G \circ F)(\boldsymbol{x}) = CA\,\boldsymbol{x} + (C\boldsymbol{b} + \boldsymbol{d})$; the map on the right acts first, and the order matters. Inverse: $\boldsymbol{x} = A^{-1}(\boldsymbol{x}' - \boldsymbol{b})$ (subtract $\boldsymbol{b}$ first); for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. If $ad - bc = 0$ some line collapses to a point and there is no inverse. $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.
- $\det A = ad - bc$ is the signed area of the parallelogram of the columns; areas are multiplied by $|\det A|$. Distances (and angles) are preserved exactly when the columns of A are *orthonormal* (length 1, pairwise dot product 0); then $A^{\mathsf{T}}A = I$, $A^{-1} = A^{\mathsf{T}}$ and $\det A = \pm 1$. Determinant ± 1 alone does not make a map rigid. Rotation by θ about O: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; about C: $\boldsymbol{c} + R_\theta(\boldsymbol{x} - \boldsymbol{c})$.
- Equation of an image: solve the map for the old coordinates and substitute them into the *old* equation (inverse substitution); or transform a parametrisation. Check one image point.
- Passive change of frame (nothing moves): new origin $\boldsymbol{o}$, orthonormal new axes as the columns of Q: $\boldsymbol{x} = \boldsymbol{o} + Q\boldsymbol{X}$, hence $\boldsymbol{X} = Q^{\mathsf{T}}(\boldsymbol{x} - \boldsymbol{o})$. To rewrite an old locus, substitute $\boldsymbol{x} = \boldsymbol{o} + Q\boldsymbol{X}$ into its equation. Always say whether the points moved (active) or the axes did (passive).

Problem 3.1 Core

Distance, foot and reflected point

After Lecture 5

In the Cartesian plane let $\ell:\ 3x+4y=5$ and $P=(6,3)$.

(a) Draw ℓ and P on equal-scale axes. Find the foot H of the perpendicular from P to ℓ by writing $H=P-\lambda\,(3,4)$ and asking that H lie on ℓ. Then check three things: H satisfies the equation, $\overrightarrow{HP}$ is perpendicular to ℓ, and PH equals the distance $d(P,\ell)$ given by the formula of Lecture 5.

(b) Find the mirror image P' of P in ℓ. Check that the midpoint of $[PP']$ lies on ℓ and that $\overrightarrow{PP'}$ is perpendicular to ℓ. On which side of ℓ is P', if $(3,4)$ points to the positive side?

(c) Compute the signed distance of P from ℓ for the oriented normal $(3,4)$. Now describe the same line by $-6x-8y=-10$ and recompute. Which of H, P', the distance and the signed distance have changed, and why?

(d) Let X be any point of ℓ. Prove directly, by expanding $\left\|\overrightarrow{PH}+\overrightarrow{HX}\right\|^2$, that $PX^2=PH^2+HX^2$. What does this identity prove about H that the perpendicularity check in (a) does not?

Problem 3.2 Core

Order matters

After Lecture 6

Let $R(x,y)=(-y,x)$ be the quarter-turn about the origin (anticlockwise) and $T(x,y)=(x+2,\ y-1)$ a translation. Both are active maps: the points move, the Cartesian frame stays; a tuple is read as a column when a matrix acts on it.

(a) For $P=(1,3)$ compute $R(P)$ and then $T(R(P))$; separately compute $T(P)$ and then $R(T(P))$. Draw the two routes on the same equal-scale axes.

(b) Find coordinate formulas for $T\circ R$ and for $R\circ T$, and write each in the form $A\boldsymbol{x}+\boldsymbol{b}$. In the notation $F\circ G$, which map is applied first? Which part of the formula is the same for the two maps, and which part differs?

(c) Show that the two final points differ by the same vector, whatever the starting point. Can $T\circ R$ and $R\circ T$ ever send a point to the same place?

(d) Undo $T\circ R$: write a formula for $(T\circ R)^{-1}$ in terms of the output coordinates (X,Y), say in which order you undid the two steps, and check it by recovering P from the final point found in (a).

Problem 3.3 Core

A shear keeps some things and changes others

After Lecture 6

Consider the shear $S(x,y)=(x+2y,\ y)$ of the Cartesian plane, and the points $O=(0,0)$, $A=(2,0)$, $B=(0,1)$.

(a) Find the images of O, A, B and of the fourth vertex $(2,1)$ of the rectangle $OACB$. Draw the rectangle and its image on equal-scale axes. Compare their areas, using a base and the corresponding height.

(b) Compare the lengths OB and $S(O)S(B)$, and the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with the angle between their images. Is S an isometry? Justify your answer without relying on the drawing.

(c) Write the image of the line $x=1$ parametrically and then as a Cartesian equation in the output coordinates (X,Y); check two of its points. Is the image parallel to the image of the line $x=0$?

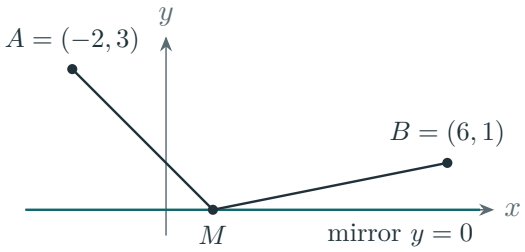
(d) Find S^{-1} and use inverse substitution to find the equation of the image of the circle $x^2+y^2=1$. Is the image the circle $X^2+Y^2=1$? Give one exact point that settles the question.

Problem 3.4 Medium

The shortest broken path

After Lecture 5

A straight mirror lies along the line $y=0$. Let $A=(-2,3)$ and $B=(6,1)$, both above the mirror. A route goes from A to a point $M=(m,0)$ of the mirror and then to B; its length is $AM+MB$.



One candidate contact point M, not claimed to be the best one. Equal units on the axes.

(a) Let B^* be the mirror image of B. Explain why $MB=MB^*$ for every point M of the mirror, and deduce that $AM+MB\geq AB^*$. State exactly when equality holds.

(b) Find the point M_0 at which the minimum is attained and the minimum length. Show that M_0 lies on the *segment* $[AB^*]$, not merely on the line through A and B^*, and explain why no other point of the mirror gives the same length.

(c) At M_0, compare the acute angles that the two parts of the route make with the mirror, using the horizontal and vertical components of $\overrightarrow{M_0A}$ and $\overrightarrow{M_0B}$.

(d) Now only the part $0\leq m\leq 2$ of the mirror may be used. Show that the bound of (a) is no longer attained. Then prove that for two points M_1, M_2 of the mirror with M_2 between M_1 and M_0, one has $AM_2+M_2B\leq AM_1+M_1B$, and deduce the shortest route for the short mirror and its length.

Problem 3.5 Medium

Calibrating an affine map

After Lecture 6

A camera is modelled by an affine map $F(\boldsymbol{x})=A\boldsymbol{x}+\boldsymbol{b}$ from the plane of a flat object to the plane of the picture, both with Cartesian coordinates; tuples are read as columns. Three marked points of the object were located in the picture:

$$F(0,0)=(1,-2),\qquad F(1,0)=(3,-1),\qquad F(0,1)=(2,-1).$$

(a) Recover $\boldsymbol{b}$ and the two columns of A, and explain why these three measurements determine F completely. Compute $F(2,-1)$.

(b) Which point of the object appears at $(4,3)$ in the picture? Give a formula for $F^{-1}(X,Y)$ valid for every output (X,Y), and check it by composing it with F.

(c) Find the equation of the image of the line $x+y=1$ by inverse substitution, and check the images of $(1,0)$ and $(0,1)$ against it.

(d) Is F an isometry? Does it preserve areas? Answer both questions from the matrix A, and reconcile the two answers.

(e) Suppose the third measurement had been $F(2,0)=(5,0)$ instead of $F(0,1)=(2,-1)$. Would the three measurements still determine F? Explain, both geometrically and in terms of the columns of A.

Problem 3.6 Medium

The same idea in three dimensions

After Lecture 6

In a right-handed orthonormal frame of space consider the active map

$$F(x, y, z) = (1 - y,\ x - 2,\ z + 3).$$

(a) Write $F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. Describe the linear part geometrically, check that the columns of A are orthonormal, and prove that F preserves the distance between any two points.

(b) Find the image of $P = (2, 1, -1)$ and of the line $P + t\,(1, 2, -1)$, $t \in \mathbb{R}$. Explain why the direction is multiplied by A while the translation $\boldsymbol{b}$ is not added to it.

(c) Find F^{-1} and use inverse substitution to find the equation of the image of the plane $x + z = 2$. Check one point of the image, and compare its normal vector with the image of the old normal $(1, 0, 1)$ under A.

(d) Find the image of the sphere $x^2 + y^2 + z^2 = 4$; state its centre and radius. Which part of F moves the centre, and which part leaves the radius alone?

Problem 3.7 Challenge

Two mirrors make a rotation

After Lectures 5 and 6

Two mirror lines pass through the origin: ℓ_0 is the x-axis, and ℓ_θ has unit direction $\boldsymbol{u}_\theta = (\cos\theta, \sin\theta)$ with $0 < \theta < \pi$. Let H_0 and H_θ be the reflections in these lines, acting on vectors; positive angles are anticlockwise, and tuples are columns when matrices act.

(a) Explain geometrically why the reflection of a vector $\boldsymbol{v}$ in ℓ_θ is $H_\theta \boldsymbol{v} = 2(\boldsymbol{v} \cdot \boldsymbol{u}_\theta)\,\boldsymbol{u}_\theta - \boldsymbol{v}$, and compare with the direction formula of Lecture 5. Apply it to $(1, 0)$ and $(0, 1)$ to find the matrix of H_θ.

(b) Using $2\cos^2\theta - 1 = \cos 2\theta$, $2\sin\theta\cos\theta = \sin 2\theta$ and $2\sin^2\theta - 1 = -\cos 2\theta$, show that $H_\theta \circ H_0$ is the rotation by 2θ about the origin. What is $H_0 \circ H_\theta$? Is there a value of θ for which the order does not matter?

(c) Take $\theta = \pi/6$ and follow the point $P = (2, 0)$ through both compositions. Draw the two images and explain the signs of their second coordinates.

(d) Replace the second mirror by the line $y = h$ with $h \neq 0$ (parallel to ℓ_0). Compute both compositions with the reflection in ℓ_0 and describe them geometrically. Why does a rotation appear in (b) and not here?

Board problems

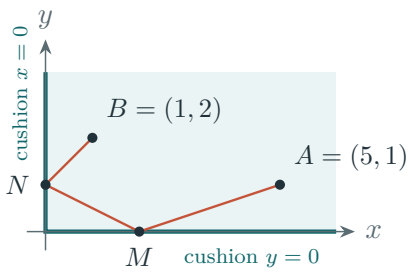
These problems are for discussion at the board. They are optional: nobody is expected to prepare all of them, and in each meeting a few are chosen. Volunteers are welcome. Work in a Cartesian frame, draw the configuration first, and say which facts from the lectures you use.

After Lecture 5

Problem 3.B1 Board

Two cushions, one straight line After Lecture 5

A billiard table occupies the quadrant $x \geq 0$, $y \geq 0$; its two cushions are the positive x-axis and the positive y-axis. A ball at $A = (5, 1)$ must bounce first off the cushion $y = 0$, at a point M, then off the cushion $x = 0$, at a point N, and then reach $B = (1, 2)$. (Ideal bounces: straight segments between the contact points.)



The table, the two endpoints and one candidate route $A \to M \to N \to B$. Equal units on the axes.

(a) Let A^* be the mirror image of A in the line $y = 0$ and B^* the mirror image of B in the line $x = 0$. Show that for every M on the line $y = 0$ and every N on the line $x = 0$,

$$AM + MN + NB = A^*M + MN + NB^* \;\geq\; A^*B^*,$$

and state when equality holds.

(b) Find the points M and N at which equality holds, check that they lie on the actual cushions (not merely on the two axes) and in the right order along the route, and give the length of the shortest route. Verify by computing AM, MN and NB separately.

(c) Now the ball must hit the cushion $x = 0$ first and the cushion $y = 0$ second. Apply the same method and explain why it does *not* produce an admissible route in this case. (You are not asked to find the shortest route of this kind.)

Problem 3.B2 Board

A perpendicular landing on a plane After Lecture 5

In a right-handed orthonormal frame of space let $\Pi : \; 2x - y + 2z = 3$ and $P = (5, 1, 6)$.

(a) Find the foot H of the perpendicular from P to Π, the distance $d(P, \Pi)$ and the mirror image P' of P in Π. Check that H satisfies the equation of Π and that H is the midpoint of $[PP']$.

(b) Prove directly, by expanding $\left\| \overrightarrow{PH} + \overrightarrow{HX} \right\|^2$, that $PX^2 = PH^2 + HX^2$ for every point X of Π, and explain why this makes H the unique closest point of the plane.

(c) The point $Q = (1, 1, 1)$ lies in Π. What do the foot and reflection formulas give for Q? If the equation is multiplied by -1, which of the answers in (a) change?

Problem 3.B3 Board

Reflect a point, then reflect a direction After Lecture 5

Let F be the reflection in the line $y = 2$. Put $P = (1, 5)$, $\boldsymbol{v} = (2, -3)$ and $Q = P + \boldsymbol{v}$.

(a) Find $F(P)$ and $F(Q)$ geometrically, using the midpoint condition. Compute the displacement $\overrightarrow{F(P)F(Q)}$.

(b) Independently reflect the direction $\boldsymbol{v}$, by keeping its component along the mirror and reversing its component across it. Compare with (a).

(c) A student applies the point formula $F(x, y) = (x,\, 4 - y)$ to the tuple $(2, -3)$ and calls the result the reflected direction. Explain the mistake. Then find the image of the whole line $P + t\boldsymbol{v}$, $t \in \mathbb{R}$.

(d) Repeat (a) and (b) for the vertical mirror $x = -1$, and write down the point formula and the direction formula for this mirror. Why does neither formula need a slope?

Problem 3.B4 Board

Parallel boundaries and a safety margin After Lecture 5

In the Cartesian plane consider the parallel lines $\ell_1 :\ 3x + 4y = 5$ and $\ell_2 :\ 3x + 4y = 15$, both oriented by the normal $(3, 4)$.

(a) Find the distance between ℓ_1 and ℓ_2. Explain why every point of ℓ_2 has the same signed distance from ℓ_1, and why multiplying the equation of ℓ_1 by -1 would change the sign of that signed distance but not the distance between the lines.

(b) A rule classifies the point $P = (1, 1)$ as being on the positive side of ℓ_1. Find the signed distance of P from ℓ_1 and the point of ℓ_1 closest to P.

(c) Prove that no displacement $\boldsymbol{e}$ with $\|\boldsymbol{e}\| < \frac{2}{5}$ can move P onto ℓ_1 or across it, and exhibit a displacement of length exactly $\frac{2}{5}$ that moves P onto ℓ_1. (This is a geometric model of a *margin*: how much a point can be perturbed before the rule changes its answer.)

Problem 3.B5 Board

When straightening the path changes After Lecture 5

A contact point $M = (m, 0)$ may be chosen anywhere on the whole line $y = 0$.

(a) Let $A = (-1, 2)$ and $B = (5, -1)$, on opposite sides of the line. Find the minimum of $AM + MB$ and all the contact points at which it is attained. Start from the triangle inequality for A, M, B directly, without reflecting anything.

(b) Now reflect B in the line $y = 0$. Does the segment from A to the reflected point cross the line? What does the reflected lower bound $AM + MB \geq AB^*$ say in this configuration, and why does it not give the answer to (a)?

(c) Put both endpoints on the line: $A = (1, 0)$ and $B = (4, 0)$. Find all the contact points that minimise $AM + MB$. What has happened to uniqueness, and where in the argument of Lecture 5 was uniqueness obtained?

After Lecture 6

Problem 3.B6 Board

Move the point or change its coordinates? After Lecture 6

Keep a fixed orthonormal Cartesian frame and the point P with coordinates $\boldsymbol{x} = (5, 3)$. A new orthonormal frame has origin O' at $\boldsymbol{o} = (2, 1)$ and axes along the columns of

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Draw the new axes in the old frame. Using $\boldsymbol{x} = \boldsymbol{o} + Q\boldsymbol{X}$, find the coordinates $\boldsymbol{X} = (X, Y)$ of the same point P in the new frame, and check by reconstructing $\boldsymbol{x}$ from them. Which way does each new axis point?

(b) By contrast, apply the *active* map $G(\boldsymbol{x}) = \boldsymbol{o} + Q\boldsymbol{x}$ to P, keeping the old frame. Which point do you obtain, and why is its coordinate pair not the pair found in (a), although the same $\boldsymbol{o}$ and Q appear in both formulas?

(c) Rewrite the old line $x + y = 4$ in the new coordinates (X, Y). Has the line moved? Say which convention, active or passive, you used.

Problem 3.B7 Board

When an affine map collapses a line After Lecture 6

Consider the map $C(x, y) = (x + y,\ 2x + 2y)$ in fixed Cartesian coordinates.

(a) Compute $C(1, 0)$ and $C(0, 1)$ and write the matrix of C. Find all the points sent to $(0, 0)$ and describe the image of the whole plane. Does C have an inverse?

(b) Find the image of the line $x + y = 3$ and the image of the line $x - y = 1$, using parametrisations with the parameter running through all of $\mathbb{R}$.

(c) A student claims: "Every affine map sends every line to a line." Correct the statement, and say which nonzero directions collapse under C.

Problem 3.B8 Board

Why an affine map keeps a midpoint After Lecture 6

Let $F(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ be an affine map of the plane; A need not be invertible. Points are identified with their coordinate columns in the fixed frame.

(a) For two points P, Q and a number $t \in \mathbb{R}$, prove that

$$F\big((1 - t)P + tQ\big) = (1 - t)F(P) + tF(Q).$$

What does this say about the image of the midpoint of $[PQ]$, and about the image of the segment $[PQ]$? Include the case $F(P) = F(Q)$.

(b) Let $P = (0, 0)$, $Q = (2, 0)$, $R = (0, 3)$ and $F(x, y) = (2x,\ y + 1)$. Find the centroid of the triangle PQR and the centroid of the image triangle, and check that one is the image of the other. Which property of the weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ makes this work in general?

(c) Does preserving midpoints imply preserving lengths? Use the same map to decide. Where exactly in the proof of (a) did you use that the two coefficients add up to 1?